

# On Non-topological Solutions of the $\mathbf{G}_2$ Chern-Simons System<sup>\*</sup>

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## Abstract

For any rank 2 of simple Lie algebra, the relativistic Chern-Simons system has the following form:

$$\begin{cases} \Delta u_1 + (\sum_{i=1}^2 K_{1i} e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i} K_{1i} e^{u_j} K_{ij}) = 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + (\sum_{i=1}^2 K_{2i} e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i} K_{2i} e^{u_j} K_{ij}) = 4\pi \sum_{j=1}^{N_2} \delta_{q_j} \end{cases} \quad \text{in } \mathbb{R}^2, \quad (0.1)$$

where  $K$  is the Cartan matrix of rank 2. There are three Cartan matrix of rank 2:  $\mathbf{A}_2$ ,  $\mathbf{B}_2$  and  $\mathbf{G}_2$ . A long-standing open problem for (0.1) is the question of the existence of non-topological solutions. In a previous paper [1], we have proven the existence of non-topological solutions for the  $\mathbf{A}_2$  and  $\mathbf{B}_2$  Chern-Simons system. In this paper, we continue to consider the  $\mathbf{G}_2$  case. We prove the existence of non-topological solutions under the condition that either  $N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j$  or

$N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j$  and  $N_1, N_2 > 1$ ,  $|N_1 - N_2| \neq 1$ . We solve this problem by a perturbation from the corresponding  $\mathbf{G}_2$  Toda system with one singular source. Combining with [1], we have proved the existence of non-topological solutions to the Chern-Simons system with Cartan matrix of rank 2.

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# 1 Introduction

## 1.1 Background

There are four types of simple non-exceptional Lie Algebra:  $\mathbf{A}_m$ ,  $\mathbf{B}_m$ ,  $\mathbf{C}_m$ , and  $\mathbf{D}_m$  which Cartan subalgebra are  $sl(m+1)$ ,  $so(2m+1)$ ,  $sp(m)$ , and  $so(2m)$  respectively. To each of them, a Toda system is associated. In geometry, solutions of Toda system is closely related to holomorphic curves in projective spaces. For example, the Toda system of type  $\mathbf{A}_m$  can be derived from the classical Plücker formulas, and any holomorphic curve gives rise to a solution  $u$  of the Toda system, whose branch points correspond to the singularities of  $u$ . Conversely, we could integrate the Toda system, and any solution  $u$  gives rise to a holomorphic curve in  $\mathbb{CP}^n$  at least locally. See [9], [21] and reference therein. It is very interesting to note that the reverse process holds globally if the domain for the equation is  $\mathbb{S}^2$  or  $\mathbb{C}$ . Any solution  $u$  of type  $\mathbf{A}_m$  Toda system on  $\mathbb{S}^2$  or  $\mathbb{C}$  could produce a global holomorphic curves into  $\mathbb{CP}^n$ . This holds even when the solution  $u$  has singularities. We refer the readers to [21] for more precise statements of these results.

In physics, the Toda system also plays an important role in non-Abelian gauge field theory. One example is the relativistic Chern-Simons model proposed by Dunne [10, 11, 12] in order to explain the physics of high critical temperature superconductivity. See also [18], [19] and [20].

The model is defined in the (2+1) Minkowski space  $\mathbb{R}^{1,2}$ , the gauge group is a compact Lie group with a semi-simple Lie algebra  $\mathcal{G}$ . The Chern-Simons Lagrangian density  $\mathcal{L}$  is defined by:

$$\mathcal{L} = -k\epsilon^{\mu\nu\rho}tr(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho) - tr((D_\mu\phi)^\dagger D^\mu\phi) - V(\phi, \phi^\dagger)$$

for a Higgs field  $\phi$  in the adjoint representation of the compact gauge group  $G$ , where the associated semi-simple Lie algebra is denoted by  $\mathcal{G}$  and the  $\mathcal{G}$ -valued gauge field  $A_\alpha$  on 2 + 1 dimensional Minkowski space  $\mathbb{R}^{1,2}$  with metric  $\text{diag}\{-1,1,1\}$ . Here  $k > 0$  is the Chern-Simons coupling parameter,  $\text{tr}$  is the trace in the matrix representation of  $\mathcal{G}$  and  $V$  is the potential energy density of the Higgs field  $V(\phi, \phi^\dagger)$  given by

$$V(\phi, \phi^\dagger) = \frac{1}{4k^2}tr(([[\phi, \phi^\dagger], \phi] - v^2\phi)^\dagger([[\phi, \phi^\dagger], \phi] - v^2\phi)),$$

where  $v > 0$  is a constant which measures either the scale of the broken symmetry or the subcritical temperature of the system.

In general, the Euler-Lagrangian equation corresponding  $\mathcal{L}$  is very difficult to study. So we restrict to consider solutions to be energy minimizers of the Lagrangian functional, and then a self-dual system of first order derivatives could be derived from minimizing the energy functional:

$$\begin{aligned} D_- \phi &= 0 \\ F_{+-} &= \frac{1}{k^2} [v^2\phi - [[\phi, \phi^\dagger], \phi], \phi^\dagger], \end{aligned} \tag{1.1}$$

where  $D_- = D_1 - iD_2$ , and  $F_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-]$  with  $A_\pm = A_1 \pm iA_2$ ,  $\partial_\pm = \partial_1 \pm i\partial_2$ . Here  $\partial_i$  and  $D_i$  are respectively the partial derivative and the gauge-covariant derivative w.r.t  $z_i$ ,  $i = 1, 2$ .

In order to find non-trivial solutions which are not algebraic solutions of  $[[\phi, \phi^\dagger], \phi] = v^2 \phi$ , Dunne [11] has considered a simplified form of the self-dual system (1.1) in which both the gauge potential  $A$  and the Higgs field  $\phi$  are algebraically restricted, for example,  $\phi$  has the following form:

$$\phi = \sum_{a=1}^r \phi^a E_a,$$

where  $r$  is the rank of the Lie algebra  $\mathcal{G}$ ,  $\{E_{\pm a}\}$  is the family of the simple root step operators (with  $E_{-a} = E_a^+$ ), and  $\phi^a$  are complex-valued functions.

In this paper, we consider the case of rank 2. Let

$$u_a = \ln |\phi^a|^2.$$

By using this ansatz, then equation (1.1) can be reduced to

$$\begin{cases} \Delta u_1 + \frac{v^4}{k^2} (\sum_{i=1}^2 K_{1i} e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i} K_{1i} e^{u_j} K_{ij}) = 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + \frac{v^4}{k^2} (\sum_{i=1}^2 K_{2i} e^{u_i} - \sum_{i=1}^2 \sum_{j=1}^2 e^{u_i} K_{2i} e^{u_j} K_{ij}) = 4\pi \sum_{j=1}^{N_2} \delta_{q_j} \end{cases} \quad \text{in } \mathbb{R}^2, \quad (1.2)$$

where  $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$  is the Cartan matrix of rank 2 of the Lie algebra  $\mathcal{G}$ ,  $\{p_1, \dots, p_{N_1}\}$  and  $\{q_1, \dots, q_{N_2}\}$  are given vortex points. For the details of the process to derive (1.2) from (1.1), we refer to [11],[27],[36] and [37]. In this paper, without loss of generality, we assume  $\frac{v^4}{k^2} = 1$ .

It is known that there are only three types of Cartan matrix of rank 2, given by

$$\mathbf{A}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{B}_2 (= \mathbf{C}_2) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad \mathbf{G}_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \quad (1.3)$$

In the previous paper [1], we have constructed non-topological solutions in the case of  $\mathbf{A}_2$  and  $\mathbf{B}_2$ . In this paper, we will construct non-topological solutions for the  $\mathbf{G}_2$  case, i.e. the following equation:

$$\begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} = 4e^{2u_1} - 2e^{2u_2} + e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + 2e^{u_2} - 3e^{u_1} = 4e^{2u_2} - 6e^{2u_1} - 3e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_2} \delta_{q_j}. \end{cases} \quad (1.4)$$

## 1.2 Previous Results

In the literature, a solution  $\mathbf{u} = (u_1, u_2)$  to system (1.2) is called a *topological* solution if  $\mathbf{u}$  satisfies

$$u_a(z) \rightarrow \ln \sum_{j=1}^2 (K^{-1})_{aj} \quad \text{as } |z| \rightarrow +\infty, \quad a = 1, 2,$$

and is called a *non-topological* solution if  $\mathbf{u}$  satisfies

$$u_a(z) \rightarrow -\infty \quad \text{as } |z| \rightarrow +\infty, \quad a = 1, 2. \quad (1.5)$$

The existence of topological solutions with arbitrary multiple vortex points was proved by Yang [37] more than fifteen years ago, not only for Cartan matrix of rank 2, but also for general Cartan matrix including  $SU(N+1)$  case,  $N \geq 1$ . However, the existence of non-topological solutions is more difficult to prove. The first result was due to Chae and Imanuvilov [4] for the  $SU(2)$  Abelian Chern-Simons equation which is obtained by letting  $u_1(z) = u_2(z) = u(z)$  in the  $\mathbf{A}_2$  system where  $u$  satisfies

$$\Delta u + e^u(1 - e^u) = 4\pi \sum_{j=1}^N \delta_{p_j} \quad \text{in } \mathbb{R}^2. \quad (1.6)$$

Equation (1.6) is the  $SU(2)$  Chern-Simons equation for the Abelian case. This relativistic Chern-Simons model was proposed by Jackiw-Weinberg [17] and Hong-Kim-Pac [16]. For the past more than twenty years, the existence and multiplicity of solutions to (1.6) with different nature (e.g. topological, non-topological, periodically constrained etc.) have been studied, see [3], [4], [5], [6], [7], [16], [22], [23], [24], [25], [29], [30], [31], [32], [33] and references therein.

In [4], Chae and Imanuvilov proved the existence of non-topological solutions for (1.6) for any vortex points  $(p_1, \dots, p_N)$ . For the question of existence of non-topological solutions for the  $\mathbf{A}_2$  system, an “answer” was given by Wang and Zhang [35] but their proof contains serious gaps. In fact they used a special solution of the Toda system as the approximate solution, but they did not have the full non-degeneracy of the linearized equation of the Toda system and their analysis for the linearized equation is incorrect. Thus, the existence of non-topological solutions has remained a long-standing open problem. Even for radially symmetric solutions (the case when all the vortices coincide), the ODE system is much more subtle than equation (1.6). The classification of radial solution is an important issue for future study as long as bubbling solutions are concerned, see [4], [5], [6], [7], [13], [22], [23], [26], [28], [31], [32], [34] in this direction.

For the rank 2 Chern-Simons system of Lie type, Huang and the second author [14, 15] studied the structure of radial solutions. Among other things, they proved the following result:

**Theorem A** If  $(u_1, u_2)$  is a radially symmetric non-topological solutions to the rank 2 Chern-Simons system of Lie type with all vortices at the origin, then

$$u_1(r) = -2\alpha_1 \log r + O(1), \quad u_2(r) = -2\alpha_2 \log r + O(1) \quad (1.7)$$

at infinity for some  $\alpha_1, \alpha_2 > 1$ . Furthermore,

$$J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1),$$

where  $J(x, y)$  is the quadratic form associated to  $K^{-1}$ .

For the existence of non-topological solutions of radially symmetric solutions, recently, in [8], Choe, Kim and the second author proved the following result:

**Theorem B** If  $(\alpha_1, \alpha_2)$  defined in (1.7) satisfies

$$\begin{aligned} -2N_1 - N_2 - 3 &< \alpha_2 - \alpha_1 < 2N_2 + N_1 + 3, \\ 2\alpha_1 + \alpha_2 &> N_1 + 2N_2 + 6 \quad \text{and} \quad \alpha_1 + 2\alpha_2 > 2N_1 + N_2 + 6, \end{aligned}$$

then the  $\mathbf{A}_2$  Chern-Simons system has a radially symmetric solution  $(u_1, u_2)$  subject to the boundary condition (1.7).

For general configuration vortices in  $\mathbb{R}^2$ , we first got the existence of non-topological solutions for the  $\mathbf{A}_2$  and  $\mathbf{B}_2$  Chern-Simons system by perturbation from the  $\mathbf{A}_2$  and  $\mathbf{B}_2$  Toda system with a singular source. In [1], we proved the following:

**Theorem C** Let  $\{p_j\}_{j=1}^{N_1}, \{q_j\}_{j=1}^{N_2} \subset \mathbb{R}^2$ . If either

$$(a) \quad N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j ;$$

or

$$(b) \quad N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j \quad \text{and} \quad N_1, N_2 > 1, \quad |N_1 - N_2| \neq 1, \quad \text{then there exists a non-}$$

topological solution  $(u_1, u_2)$  of the  $\mathbf{A}_2$  and  $\mathbf{B}_2$  Chern-Simons system respectively.

### 1.3 Main Results

In this paper, we continue our work on the rank 2 Chern-Simons system. We consider the remaining  $\mathbf{G}_2$  Chern-Simons system. We give an affirmative answer to the existence of non-topological solutions for the system with Cartan matrix  $\mathbf{G}_2$ . Our main theorem can be stated as follows.

**Theorem 1.1.** *Let  $\{p_j\}_{j=1}^{N_1}, \{q_j\}_{j=1}^{N_2} \subset \mathbb{R}^2$ . If either*

$$(a) \quad N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j ;$$

or

$$(b) \quad N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j \quad \text{and} \quad N_1, N_2 > 1, \quad |N_1 - N_2| \neq 1, \quad \text{then there exists a non-}$$

topological solution  $(u_1, u_2)$  of problem (1.4).

**Remark 1.1.** *Note that if  $N_1 = 0$  or  $N_2 = 0$ , by translation, assumption (a) in Theorem 1.1 is always satisfied.*

Non-topological solutions play very important role in the bubbling analysis of solutions to (1.2). Therefore, our result is only the first step towards understanding the solution structure of non-topological solution of (1.2). For further study on non-topological solutions for the Abelian case, we refer to [5] and [7].

We will prove Theorem 1.1 in three cases which we describe below:

$$\textbf{Assumption (i):} \quad \sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j, \quad N_1 = N_2; \quad (1.8)$$

$$\begin{aligned} \textbf{Assumption (ii):} \quad & N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j, \quad N_1 \neq N_2, \quad N_1, N_2 \neq 1 \\ & \text{or } N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j, \quad |N_1 - N_2| \neq 1, \quad N_1, N_2 > 1; \end{aligned} \quad (1.9)$$

$$\textbf{Assumption (iii):} \quad N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j, \quad N_1 \neq N_2, \quad N_1 = 1 \text{ or } N_2 = 1. \quad (1.10)$$

If we can prove the existence of non-topological solutions under the above three assumptions separately, then it is easy to see that Theorem 1.1 is proved. So in the following, we will prove the theorem under the three assumptions respectively.

## 1.4 Sketch of the Proof

In the following, we will outline the sketch of our proof. We follow exactly the same idea as in the proof for the  $\mathbf{A}_2$  and  $\mathbf{B}_2$  case.

As in [1], we will view equation (1.4) as a small perturbation of the  $\mathbf{G}_2$  Toda system with a singular source.

After a suitable scaling transformation, the system (1.4) is transformed to

$$\begin{cases} \Delta \tilde{U}_1 + \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 e^{2\tilde{U}_1 - \tilde{U}_2} \\ = 2\varepsilon^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^4 e^{4\tilde{U}_1 - 2\tilde{U}_2} - \varepsilon^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 e^{\tilde{U}_2 - \tilde{U}_1}, \\ \Delta \tilde{U}_2 + \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 e^{2\tilde{U}_2 - 3\tilde{U}_1} \\ = 2\varepsilon^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^4 e^{4\tilde{U}_2 - 6\tilde{U}_1} - 3\varepsilon^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 e^{\tilde{U}_2 - \tilde{U}_1}. \end{cases} \quad (1.11)$$

When  $\varepsilon = 0$ , we obtain the following limiting system

$$\begin{cases} \Delta \tilde{U}_1 + |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} = 0 \quad \text{in } \mathbb{R}^2 \\ \Delta \tilde{U}_2 + |z|^{2N_2} e^{2\tilde{U}_2 - 3\tilde{U}_1} = 0 \quad \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} < +\infty, \quad \int_{\mathbb{R}^2} |z|^{2N_2} e^{2\tilde{U}_2 - 3\tilde{U}_1} < +\infty \end{cases} \quad (1.12)$$

which is the  $\mathbf{G}_2$  Toda system with a single source at the origin.

An immediate problem is the classification and non-degeneracy of the above system. In [21], Lin, Wei and Ye obtained the classification and non-degeneracy results of the  $SU(N+1)$  Toda system with singular sources. In [2], we use the results of [21] to obtain a complete classification and non-degeneracy of the  $\mathbf{G}_2$  Toda system (1.12). In fact, the

Toda system with  $\mathbf{G}_2$  can be embedded into the  $\mathbf{A}_6$  Toda system under suitable group action. The solutions to (1.12) depend on fourteen parameters

$$(\mathbf{a}, \lambda) = (c_{43,1}, c_{43,2}, c_{52,1}, c_{52,2}, c_{53,1}, c_{53,2}, c_{54,1}, c_{54,2}, c_{61,1}, c_{61,2}, c_{62,1}, c_{62,2}, \lambda_4, \lambda_5).$$

The dimension of the linearized operator is **fourteen**.

The main difficulty of dealing with system is the large dimension of kernels. There are no explicit formula for the coefficients, except in the case  $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j$ ,  $N_1 = N_2$  which can be considered as the reminiscent of the  $SU(2)$  scalar equation. To get over this difficulty, we make use of the two scaling parameters for solutions of the Toda system and introduce two more free parameters. Instead of solving the coefficient matrices for fixed scaling parameters, we only need to compute the two matrices in front of the two free scaling parameters we introduce.

Now let us be more specific. The term of order  $O(\varepsilon)$  will satisfy (2.28) and (2.29) in Section 2.4. The  $O(\varepsilon^2)$  term will satisfy (2.30). In this  $O(\varepsilon^2)$  term  $\psi$ , we introduce **two free parameters**  $\xi_1, \xi_2$  which play an important role in our proof. See Section 5 and 6. At last the solution we find will be of this form

$$\tilde{U} = \tilde{U}_{\mathbf{b}} + \varepsilon \Psi + \varepsilon^2 \psi + \varepsilon^2 v, \quad (1.13)$$

where  $\tilde{U}_{\mathbf{b}} = (\tilde{U}_{1,\mathbf{b}}, \tilde{U}_{2,\mathbf{b}})$  is the solution of (1.12), and  $\mathbf{b}$  denote the parameters  $(\lambda, \mathbf{a})$  for simplicity of notations. In order to solve in  $v$ , we need to solve a linearized problem:

$$\begin{cases} \Delta \phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\phi_1 - \phi_2) = f_1 \\ \Delta \phi_2 + |z|^{2N_1} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\phi_2 - 3\phi_1) = f_2 \end{cases} \quad (1.14)$$

where  $f_1$  and  $f_2$  are explicitly given. Due to the existence of the kernel of the linearized equation,  $(f_1, f_2)$  must satisfy some extra condition in order to have a solution. See Lemma 2.2 for the necessary and sufficient condition. After that, we use the Liapunov-Schmidt reduction method to solve the nonlinear equation. It turns out that we can choose the perturbation  $\mathbf{a}$  and  $\lambda$  such that we can get the solution.

Now we comment on the technical conditions. In the proof, we will choose  $(\lambda_4, \lambda_5)$  first, depending on the assumptions. In general, the reduced problem for  $\mathbf{a}$  has the following form

$$\frac{1}{\varepsilon} \mathbf{B}\mathbf{a} + \frac{1}{\varepsilon} \mathbf{A}\mathbf{a} \cdot \mathbf{a} + \mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) + \mathbf{a}_0 = O(\varepsilon), \quad (1.15)$$

where  $\mathbf{A}, \mathbf{B}, \mathbf{Q}$  are matrices of size  $12 \times 12$ , and  $\mathbf{a}_0 \in \mathbb{R}^{12}$ . Furthermore, the matrix  $\mathbf{Q}$  can be decomposed into

$$\mathbf{Q} = \xi_1 \mathbf{Q}_1 + \xi_2 \mathbf{Q}_2 + \mathcal{T} \quad (1.16)$$

where  $\xi_1$  and  $\xi_2$  are two free parameters. As we said before, we shall not attempt to compute the matrices  $\mathbf{A}, \mathbf{B}$  and  $\mathcal{T}$ . Instead we focus on the two matrices  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$ . All we need to show is that at least of one of these two matrices is non-degenerate.

In case (a) of Theorem 1.1, i.e.  $N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j$ , by a shift of origin, we may

assume that  $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$ . In this case, the  $\varepsilon$ -term  $\varepsilon \Psi$  vanishes and both  $\mathbf{A}$  and

$\mathbf{B}$  vanish. If  $N_1 \neq N_2$ ,  $N_1, N_2 \neq 1$  or  $N_1 = N_2$ , then  $\mathbf{a}_0$  vanishes and we obtain a reduced problems (in terms of  $\mathbf{a}$ ) as follows:

$$\mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) = O(\varepsilon). \quad (1.17)$$

If  $N_1 \neq N_2$ ,  $N_1 = 1$ , or  $N_2 = 1$ , we use a different  $O(\varepsilon^2)$  approximation  $\psi$  in (1.13) and we obtain the reduced problem as follows:

$$\mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) + \mathbf{a}_0 = O(\varepsilon). \quad (1.18)$$

See Section 6. In both cases, we can show that the matrix  $\mathbf{Q}$  is non-degenerate and (1.15) can be solved by contraction mapping.

The case (b) is considerably more difficult. Since  $N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j$ , the  $\varepsilon$ - term exists and presents great difficulty in solving the reduced problem (1.15) in  $\mathbf{a}$ . To show that  $\mathbf{B} = 0$ , we need  $|N_1 - N_2| \neq 1$ . To show that the  $\mathbf{a}_0$  term vanishes, we need  $N_1, N_2 > 1$ . In this case the reduced problem now takes the form

$$\frac{1}{\varepsilon} \mathbf{A}\mathbf{a} \cdot \mathbf{a} + \mathbf{Q}\mathbf{a} + O(|\mathbf{a}|^2) = O(\varepsilon). \quad (1.19)$$

where  $\mathbf{Q}$  has the form of (1.16). By choosing large  $\xi_1$  and  $\xi_2 = 0$ , we can solve (1.19) such that  $|\mathbf{a}| \leq O(\varepsilon)$ .

In summary, the technical condition we have imposed is to make sure that  $\mathbf{B} = 0$  and that the quadratic term  $\frac{1}{\varepsilon} \mathbf{A}\mathbf{a} \cdot \mathbf{a}$  and the  $O(1)$  term can not coexist.

The organization of the paper is the following. In Section 2, we present several important preliminaries of analysis. We first formulate our problem in terms of the functional equations (Section 2.1). Then we apply the classification and nondegeneracy result of  $\mathbf{G}_2$  Toda system (Section 2.2). In Section 2.3, we establish the invertibility properties of the linearized operator. Finally we obtain the next two orders  $O(\varepsilon)$  and  $O(\varepsilon^2)$  in Section 2.4. In Section 3, we solve a projected nonlinear problem based on the preliminary results in Section 2. In Section 4, we prove our main theorem under the **Assumption (i)**. In Section 5, we prove the theorem under the **Assumption (ii)**. In Section 6, we prove the theorem under the **Assumption (iii)**.

## 2 Preliminaries

In this section, we consider the following  $\mathbf{G}_2$  system in  $\mathbb{R}^2$ :

$$\left\{ \begin{array}{l} \Delta u_1 + 2e^{u_1} - e^{u_2} = 4e^{2u_1} - 2e^{2u_2} + e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_1} \delta_{p_j} \\ \Delta u_2 + 2e^{u_2} - 3e^{u_1} = 4e^{2u_2} - 6e^{2u_1} - 3e^{u_1+u_2} + 4\pi \sum_{j=1}^{N_2} \delta_{q_j}. \end{array} \right. \quad (2.1)$$



## 2.1 Functional Formulation of the Problem

Defining

$$u_1 = \sum_{j=1}^{N_1} \ln |z - p_j|^2 + \tilde{u}_1, \quad u_2 = \sum_{j=1}^{N_2} \ln |z - q_j|^2 + \tilde{u}_2,$$

and  $z = \frac{\tilde{z}}{\varepsilon}$ , and let  $U_i$  and  $\tilde{U}_i$  to be

$$\tilde{u}_1(z) = U_1(\tilde{z}) + (2N_1 + 2) \ln \varepsilon, \quad \tilde{u}_2(z) = U_2(\tilde{z}) + (2N_2 + 2) \ln \varepsilon,$$

and

$$\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{pmatrix} = \mathbf{G}_2^{-1} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

where  $\mathbf{G}_2$  is the Cartan matrix  $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ . Then  $(\tilde{U}_1, \tilde{U}_2)$  will satisfy

$$\begin{cases} \Delta \tilde{U}_1 + \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 e^{2\tilde{U}_1 - \tilde{U}_2} \\ = 2\varepsilon^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^4 e^{4\tilde{U}_1 - 2\tilde{U}_2} - \varepsilon^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 e^{\tilde{U}_2 - \tilde{U}_1}, \\ \Delta \tilde{U}_2 + \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 e^{2\tilde{U}_2 - 3\tilde{U}_1} \\ = 2\varepsilon^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^4 e^{4\tilde{U}_2 - 6\tilde{U}_1} - 3\varepsilon^2 \Pi_{j=1}^{N_2} |\tilde{z} - \varepsilon q_j|^2 \Pi_{j=1}^{N_1} |\tilde{z} - \varepsilon p_j|^2 e^{\tilde{U}_2 - \tilde{U}_1}. \end{cases} \quad (2.2)$$

From now on, we shall work with (2.2). For simplicity of notations, we still denote the variable by  $z$  instead of  $\tilde{z}$ .

## 2.2 First Approximate Solution

When  $\varepsilon = 0$ , (2.2) becomes

$$\begin{cases} \Delta \tilde{U}_1 + |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} = 0 \\ \Delta \tilde{U}_2 + |z|^{2N_2} e^{2\tilde{U}_2 - 3\tilde{U}_1} = 0 \\ \int_{\mathbb{R}^2} |z|^{2N_1} e^{2\tilde{U}_1 - \tilde{U}_2} < \infty, \quad \int_{\mathbb{R}^2} |z|^{2N_2} e^{2\tilde{U}_2 - 3\tilde{U}_1} < \infty \end{cases} \quad (2.3)$$

For this system, we have gotten the classification and non-degeneracy result in [2]. From Theorem 2.1 in [2], we see that all the solutions of (2.3) depend on fourteen parameters  $(c_{43}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62}, \lambda_4, \lambda_5) \in \mathbb{C}^6 \times (\mathbb{R}^+)^2$ , and all the solutions of (2.3) are of the form

$$e^{-\tilde{U}_1} = 2(\lambda_0 + \sum_{i=1}^6 \lambda_i |P_i(z)|^2), \quad (2.4)$$

where

$$P_i(z) = z^{\mu_1 + \dots + \mu_i} + \sum_{j=0}^{i-1} c_{ij} z^{\mu_1 + \dots + \mu_j}, \quad (2.5)$$

$$\mu_1 = \mu_3 = \mu_4 = \mu_6 = N_1 + 1, \quad \mu_2 = \mu_5 = N_2 + 1,$$

and

$$\lambda_0 = \frac{1}{(2^{11-\frac{1}{2}}\mu_1^2\mu_2(\mu_1+\mu_2)^2(2\mu_1+\mu_2)^2(3\mu_1+\mu_2)(3\mu_1+2\mu_2))^2\lambda_4\lambda_5}, \quad (2.6)$$

$$\lambda_1 = \frac{1}{(2^7\mu_1\mu_2(\mu_1+\mu_2)^2(2\mu_1+\mu_2)(3\mu_1+2\mu_2))^2\lambda_5}, \quad (2.7)$$

$$\lambda_2 = \frac{1}{(2^7\mu_1^2\mu_2(\mu_1+\mu_2)(2\mu_1+\mu_2)(3\mu_1+\mu_2))^2\lambda_4}, \quad (2.8)$$

$$\lambda_3 = \frac{1}{(2^3\mu_1(\mu_1+\mu_2)(2\mu_1+\mu_2))^2}, \quad (2.9)$$

$$\lambda_6 = (2^{3+\frac{1}{2}}\mu_1\mu_2(\mu_1+\mu_2))^2\lambda_4\lambda_5. \quad (2.10)$$

See Theorem 2.1 in [2].

We denote by

$$(\lambda, \mathbf{a}) = (\lambda_4, \lambda_5, c_{43,1}, c_{43,2}, c_{52,1}, c_{52,2}, c_{53,1}, c_{53,2}, c_{54,1}, c_{54,2}, c_{61,1}, c_{61,2}, c_{62,1}, c_{62,2}). \quad (2.11)$$

When  $\mathbf{a} = 0$ , the radially symmetric solution of (2.3) can be expressed as follows:

$$e^{-\tilde{U}_{1,0}} := \rho_{1,G}^{-1} = 2\rho_1^{-1}, \quad (2.12)$$

$$e^{-\tilde{U}_{2,0}} := \rho_{2,G}^{-1} = 4\rho_2^{-1} \quad (2.13)$$

and

$$\begin{aligned} \rho_1^{-1} &= \lambda_0 + \lambda_1 r^{2\mu_1} + \lambda_2 r^{2(\mu_1+\mu_2)} + \lambda_3 r^{2(2\mu_1+\mu_2)} + \lambda_4 r^{2(3\mu_1+\mu_2)} + \lambda_5 r^{2(3\mu_1+2\mu_2)} + \lambda_6 r^{2(4\mu_1+2\mu_2)}, \\ \rho_2^{-1} &= 4 \left[ \lambda_0 \mu_1^2 \lambda_1 + r^{4(\mu_1+\mu_2)} (4r^{2\mu_1} (\lambda_0 \lambda_6 (2\mu_1 + \mu_2)^2 + \lambda_1 \lambda_5 (\mu_1 + \mu_2)^2) + \lambda_0 \lambda_5 (3\mu_1 + 2\mu_2)^2 \right. \\ &\quad + r^{4\mu_1} (\lambda_1 \lambda_6 (3\mu_1 + 2\mu_2)^2 + \mu_1^2 \lambda_3 \lambda_4) + \mu_1^2 \lambda_2 (\lambda_3 + 4\lambda_4 r^{2\mu_1})) \\ &\quad + r^{2\mu_2} (\lambda_0 (\lambda_2 (\mu_1 + \mu_2)^2 + \lambda_3 (2\mu_1 + \mu_2)^2 r^{2\mu_1} + \lambda_4 (3\mu_1 + \mu_2)^2 r^{4\mu_1}) \\ &\quad + \lambda_1 r^{2\mu_1} (\mu_2^2 \lambda_2 + \lambda_3 (\mu_1 + \mu_2)^2 r^{2\mu_1} + \lambda_4 (2\mu_1 + \mu_2)^2 r^{4\mu_1})) \\ &\quad + r^{6(\mu_1+\mu_2)} (r^{2\mu_1} (\lambda_2 \lambda_6 (3\mu_1 + \mu_2)^2 + \lambda_3 \lambda_5 (\mu_1 + \mu_2)^2) + \lambda_2 \lambda_5 (2\mu_1 + \mu_2)^2 \\ &\quad \left. + r^{4\mu_1} (\lambda_3 \lambda_6 (2\mu_1 + \mu_2)^2 + \mu_2^2 \lambda_4 \lambda_5) + \lambda_4 \lambda_6 (\mu_1 + \mu_2)^2 r^{6\mu_1} + \mu_1^2 \lambda_5 \lambda_6 r^{12\mu_1+8\mu_2}) \right], \quad (2.14) \end{aligned}$$

where  $\mu_1 = N_1 + 1$ ,  $\mu_2 = N_2 + 1$  and  $\lambda_i$  are defined before.

Observe that the radial solution  $(\tilde{U}_{1,0}, \tilde{U}_{2,0})$  depends on two scaling parameters  $(\lambda_4, \lambda_5)$ . Later we shall choose  $(\lambda_4, \lambda_5)$  in different ways.

Next we have the following non-degeneracy result:

**Lemma 2.1.** (Non-degeneracy) *The previous solutions of (2.3) are non-degenerate, i.e., the set of solutions corresponding to the linearized operator at  $(\tilde{U}_{1,0}, \tilde{U}_{2,0})$  is exactly fourteen dimensional. More precisely, if  $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$  satisfies  $|\phi(z)| \leq C(1 + |z|)^\alpha$  for some  $0 \leq \alpha < 1$ , and*

$$\begin{cases} \Delta \phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\phi_1 - \phi_2) = 0 \\ \Delta \phi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\phi_2 - 3\phi_1) = 0, \end{cases} \quad (2.15)$$

then  $\phi$  belongs to the following linear space  $\mathcal{K}$ : the span of

$$\{Z_{\lambda_4}, Z_{\lambda_5}, Z_{c_{43,1}}, Z_{c_{43,2}}, Z_{c_{52,1}}, Z_{c_{52,2}}, Z_{c_{53,1}}, Z_{c_{53,2}}, Z_{c_{54,1}}, Z_{c_{54,2}}, Z_{c_{61,1}}, Z_{c_{61,2}}, Z_{c_{62,1}}, Z_{c_{62,2}}\},$$

where

$$\begin{aligned} Z_{\lambda_4} &= \begin{pmatrix} Z_{\lambda_4,1} \\ Z_{\lambda_4,2} \end{pmatrix} = \begin{pmatrix} \partial_{\lambda_4} \tilde{U}_{1,0} \\ \partial_{\lambda_4} \tilde{U}_{2,0} \end{pmatrix}, \quad Z_{\lambda_5} = \begin{pmatrix} Z_{\lambda_5,1} \\ Z_{\lambda_5,2} \end{pmatrix} = \begin{pmatrix} \partial_{\lambda_5} \tilde{U}_{1,0} \\ \partial_{\lambda_5} \tilde{U}_{2,0} \end{pmatrix}, \\ Z_{c_{43,i}} &= \begin{pmatrix} Z_{c_{43,i},1} \\ Z_{c_{43,i},2} \end{pmatrix} = \begin{pmatrix} \partial_{c_{43,i}} \tilde{U}_{1,0} \\ \partial_{c_{43,i}} \tilde{U}_{2,0} \end{pmatrix}, \quad Z_{c_{52,i}} = \begin{pmatrix} Z_{c_{52,i},1} \\ Z_{c_{52,i},2} \end{pmatrix} = \begin{pmatrix} \partial_{c_{52,i}} \tilde{U}_{1,0} \\ \partial_{c_{52,i}} \tilde{U}_{2,0} \end{pmatrix}, \\ Z_{c_{53,i}} &= \begin{pmatrix} Z_{c_{53,i},1} \\ Z_{c_{53,i},2} \end{pmatrix} = \begin{pmatrix} \partial_{c_{53,i}} \tilde{U}_{1,0} \\ \partial_{c_{53,i}} \tilde{U}_{2,0} \end{pmatrix}, \quad Z_{c_{54,i}} = \begin{pmatrix} Z_{c_{54,i},1} \\ Z_{c_{54,i},2} \end{pmatrix} = \begin{pmatrix} \partial_{c_{54,i}} \tilde{U}_{1,0} \\ \partial_{c_{54,i}} \tilde{U}_{2,0} \end{pmatrix}, \\ Z_{c_{61,i}} &= \begin{pmatrix} Z_{c_{61,i},1} \\ Z_{c_{61,i},2} \end{pmatrix} = \begin{pmatrix} \partial_{c_{61,i}} \tilde{U}_{1,0} \\ \partial_{c_{61,i}} \tilde{U}_{2,0} \end{pmatrix}, \quad Z_{c_{62,i}} = \begin{pmatrix} Z_{c_{62,i},1} \\ Z_{c_{62,i},2} \end{pmatrix} = \begin{pmatrix} \partial_{c_{62,i}} \tilde{U}_{1,0} \\ \partial_{c_{62,i}} \tilde{U}_{2,0} \end{pmatrix}, \end{aligned}$$

for  $i = 1, 2$ .

For a proof, we refer Corollary 2.3 in [2]. For the reference of computation later, we want to write down the explicit expression of all kernel functions of the linearized equation. For the simplicity of notations, we use  $\tilde{U}_{1,\lambda_4}, \dots$  etc to denote  $\partial_{\lambda_4} \tilde{U}_1, \dots$  etc.

$$\begin{aligned} \tilde{U}_{1,\lambda_4} &= \rho_1 \left( r^{2(3\mu_1+\mu_2)} + 2^7 \lambda_5 r^{4(2\mu_1+\mu_2)} \mu_1^2 \mu_2^2 (\mu_1 + \mu_2)^2 - \frac{r^{2(\mu_1+\mu_2)}}{2^{14} \lambda_4^2 \mu_1^4 \mu_2^2 (\mu_1 + \mu_2)^2 (2\mu_1 + \mu_2)^2 (3\mu_1 + \mu_2)^2} \right. \\ &\quad \left. - \frac{1}{2^{21} \lambda_4^2 \lambda_5 \mu_1^4 \mu_2^2 (\mu_1 + \mu_2)^4 (2\mu_1 + \mu_2)^4 (3\mu_1 + \mu_2)^2 (3\mu_1 + 2\mu_2)^2} \right), \\ \tilde{U}_{2,\lambda_4} &= \frac{4\rho_2}{2^{25} \mu_2^4 (\mu_1 + \mu_2)^8} \left[ - \frac{1}{\lambda_4^2 \lambda_5^2 (2\mu_1 + \mu_2)^6 (3\mu_1 + \mu_2)^2 (3\mu_1^2 + 2\mu_1 \mu_2)^4} \right. \\ &\quad + \frac{2^{21} \mu_2^2 \lambda_5 (\mu_1 + \mu_2)^6 r^{6(\mu_1+\mu_2)}}{\mu_1^4 \lambda_4^2 (3\mu_1 + \mu_2)^2} \\ &\quad \times \left( 2^{14} \mu_1^4 \mu_2^4 \lambda_4^2 (\mu_1 + \mu_2)^2 (3\mu_1 + \mu_2)^2 r^{4\mu_1} (256 \mu_1^2 \lambda_4 (\mu_1 + \mu_2)^4 r^{2\mu_1} + 3) - 1 \right) \\ &\quad + \frac{3 \times 2^{14} \mu_2^2 (\mu_1 + \mu_2)^4 r^{4(\mu_1+\mu_2)} (2^{14} \mu_1^4 \mu_2^2 \lambda_4^2 (\mu_1 + \mu_2)^2 (2\mu_1 + \mu_2)^2 (3\mu_1 + \mu_2)^2 r^{4\mu_1} - 1)}{\mu_1^4 \lambda_4^2 (2\mu_1 + \mu_2)^4 (3\mu_1 + \mu_2)^2} \\ &\quad + \frac{2(\mu_1 + \mu_2)^2 r^{2\mu_2} \left( - \frac{(\mu_1+\mu_2)^2}{\lambda_4^3 (2\mu_1+\mu_2)^6 (3\mu_1+\mu_2)^4} - \frac{192 \mu_1^2 \mu_2^2 r^{2\mu_1}}{\lambda_4^2 (2\mu_1+\mu_2)^4 (3\mu_1+\mu_2)^2} + 2^{20} \mu_1^6 \mu_2^2 (\mu_1 + \mu_2)^2 r^{6\mu_1} \right)}{\mu_1^8 \lambda_5 (3\mu_1 + 2\mu_2)^2} \\ &\quad \left. + 2^{42} \mu_1^4 \mu_2^6 \lambda_5^2 (\mu_1 + \mu_2)^{10} r^{12\mu_1+8\mu_2} \right], \end{aligned}$$

$$\begin{aligned}
\tilde{U}_{1,\lambda_5} &= \rho_1 \left[ r^{6\mu_1+4\mu_2} + 2^7 \lambda_4 r^{4(2\mu_1+\mu_2)} \mu_1^2 \mu_2^2 (\mu_1 + \mu_2)^2 - \frac{r^{2\mu_1}}{2^{14} \lambda_5^2 \mu_1^2 \mu_2^2 (\mu_1 + \mu_2)^4 (2\mu_1 + \mu_2)^2 (3\mu_1 + 2\mu_2)^2} \right. \\
&\quad \left. - \frac{1}{2^{21} \lambda_4 \lambda_5^2 \mu_1^4 \mu_2^2 (\mu_1 + \mu_2)^4 (2\mu_1 + \mu_2)^4 (3\mu_1 + \mu_2)^2 (3\mu_1 + 2\mu_2)^2} \right], \\
\tilde{U}_{2,\lambda_5} &= 4\rho_2 \left[ 3\mu_2^2 \lambda_4 r^{10\mu_1+6\mu_2} + 2^8 \mu_1^4 \mu_2^2 \lambda_4 \lambda_5 (\mu_1 + \mu_2)^2 r^{12\mu_1+8\mu_2} + 2^7 \mu_1^2 \mu_2^2 \lambda_4^2 (\mu_1 + \mu_2)^4 r^{6(2\mu_1+\mu_2)} \right. \\
&\quad + \frac{3r^{8\mu_1+6\mu_2}}{2^7 \mu_1^2 (2\mu_1 + \mu_2)^2} - \frac{\lambda_4 r^{2(3\mu_1+\mu_2)}}{2^{14} \mu_1^2 \mu_2^2 \lambda_5^2 (\mu_1 + \mu_2)^4 (3\mu_1 + 2\mu_2)^2} \\
&\quad - \frac{3r^{2(2\mu_1+\mu_2)}}{2^{21} \mu_1^4 \mu_2^2 \lambda_5^2 (\mu_1 + \mu_2)^4 (2\mu_1 + \mu_2)^4 (3\mu_1 + 2\mu_2)^2} \\
&\quad - \frac{r^{2\mu_2}}{2^{35} \mu_1^8 \mu_2^4 \lambda_4^2 \lambda_5^2 (\mu_1 + \mu_2)^4 (2\mu_1 + \mu_2)^6 (3\mu_1 + \mu_2)^4 (3\mu_1 + 2\mu_2)^2} \\
&\quad \left. - \frac{1}{2^{34} \mu_1^6 \mu_2^4 \lambda_4 (\mu_1 + \mu_2)^8 (3\mu_1 + \mu_2)^2} \left( \frac{\mu_1^2}{\lambda_5^3 (2\mu_1 + \mu_2)^6 (3\mu_1 + 2\mu_2)^4} + \frac{3 \cdot 2^6 \mu_2^2 (\mu_1 + \mu_2)^2 r^{2(\mu_1+\mu_2)}}{\lambda_5^2 (2\mu_1 + \mu_2)^4 (3\mu_1 + 2\mu_2)^2} \right. \right. \\
&\quad \left. \left. - 2^{20} \mu_1^2 \mu_2^2 (\mu_1 + \mu_2)^6 r^{6(\mu_1+\mu_2)} \right) \right],
\end{aligned}$$

$$\begin{aligned}
\tilde{U}_{1,c_{43},1} &= \rho_1 \left( \lambda_4 r^{5\mu_1+2\mu_2} + \frac{\lambda_6 (2\mu_1 + \mu_2) r^{7\mu_1+4\mu_2}}{2\mu_2} + \frac{\lambda_1 (\mu_1 + \mu_2) r^{\mu_1}}{2(3\mu_1 + \mu_2)} \right. \\
&\quad \left. + \frac{\lambda_3 (\mu_1 + \mu_2) (2\mu_1 + \mu_2) r^{3\mu_1+2\mu_2}}{2\mu_2 (3\mu_1 + \mu_2)} \right) \cos \mu_1 \theta, \\
\tilde{U}_{2,c_{43},1} &= \frac{4\rho_2 r^{\mu_1+2\mu_2}}{2\mu_2 (3\mu_1 + \mu_2)} \left[ (\mu_1 + \mu_2)^2 (\lambda_0 \lambda_3 (2\mu_1 + \mu_2)^2 + \lambda_1 \mu_2^2 \lambda_2) \right. \\
&\quad + r^{4\mu_1} \left( 2r^{2\mu_2} (\lambda_0 \lambda_6 (3\mu_1 + \mu_2) (3\mu_1 + 2\mu_2) (2\mu_1 + \mu_2)^2 \right. \\
&\quad + \mu_2 (\lambda_1 \lambda_5 (\mu_1 + \mu_2)^2 (3\mu_1 + 2\mu_2) + 2\mu_1^2 \lambda_2 \lambda_4 (3\mu_1 + \mu_2))) \\
&\quad + 3\lambda_1 \mu_2 \lambda_4 (\mu_1 + \mu_2) (2\mu_1 + \mu_2) (3\mu_1 + \mu_2) \Big) \\
&\quad + 2\mu_2 (2\mu_1 + \mu_2) r^{2\mu_1} (\lambda_0 \lambda_4 (3\mu_1 + \mu_2)^2 + \lambda_1 \lambda_3 (\mu_1 + \mu_2)^2) \\
&\quad + (2\mu_1 + \mu_2) r^{2(3\mu_1+\mu_2)} \left( 2(\mu_1 + \mu_2) (\lambda_1 \lambda_6 (3\mu_1 + 2\mu_2)^2 \right. \\
&\quad + \mu_1^2 \lambda_3 \lambda_4) + (2\mu_1 + \mu_2) r^{2\mu_2} (\lambda_2 \lambda_6 (3\mu_1 + \mu_2)^2 + \lambda_3 \lambda_5 (\mu_1 + \mu_2)^2) \Big) \\
&\quad + 2(\mu_1 + \mu_2) (3\mu_1 + \mu_2) r^{4(2\mu_1+\mu_2)} (\lambda_3 \lambda_6 (2\mu_1 + \mu_2)^2 + \mu_2^2 \lambda_4 \lambda_5) \\
&\quad \left. + 3\mu_2 \lambda_4 \lambda_6 (\mu_1 + \mu_2) (2\mu_1 + \mu_2) (3\mu_1 + \mu_2) r^{10\mu_1+4\mu_2} \right] \cos \mu_1 \theta,
\end{aligned}$$

$$\begin{aligned}
\tilde{U}_{1,c52,1} &= \frac{\rho_1 r^{2\mu_1+\mu_2}}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)} \left[ (r^{2\mu_1}(\mu_1\lambda_4(3\mu_1+\mu_2) + (3\mu_1+2\mu_2)r^{2\mu_2}(\lambda_5(\mu_1+\mu_2) \right. \\
&\quad \left. - 2\lambda_6(3\mu_1+\mu_2)r^{2\mu_1})) - \mu_1\lambda_3(\mu_1+\mu_2)) \right] \cos(2\mu_1+\mu_2)\theta, \\
\tilde{U}_{2,c52,1} &= \frac{4\rho_2 r^{2\mu_1+\mu_2}}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)} \left[ r^{2\mu_2} \left( -2(3\mu_1+2\mu_2)r^{2\mu_1}(2\lambda_0\lambda_6(2\mu_1+\mu_2)^2(3\mu_1+\mu_2) \right. \right. \\
&\quad \left. - \lambda_1\mu_2\lambda_5(\mu_1+\mu_2)^2) + \lambda_0\lambda_5(\mu_1+\mu_2)^2(3\mu_1+2\mu_2)^2 \right. \\
&\quad \left. - 2(\mu_1+\mu_2)(3\mu_1+\mu_2)r^{4\mu_1}(\lambda_1\lambda_6(3\mu_1+2\mu_2)^2 + \mu_1^2\lambda_3\lambda_4) \right. \\
&\quad \left. + \mu_1^2\lambda_2(\lambda_3(\mu_1+\mu_2)^2 - 2\mu_2\lambda_4(3\mu_1+\mu_2)r^{2\mu_1})) + \mu_1^2(\lambda_0\lambda_4(3\mu_1+\mu_2)^2 + \lambda_1\lambda_3(\mu_1+\mu_2)^2) \right. \\
&\quad \left. + \mu_1r^{4(\mu_1+\mu_2)} \left( -2(3\mu_1+2\mu_2)(\lambda_2\lambda_6(3\mu_1+\mu_2)^2 + \lambda_3\lambda_5(\mu_1+\mu_2)^2) \right. \right. \\
&\quad \left. - 2(\mu_1+\mu_2)r^{2\mu_1}(\lambda_3\lambda_6(2\mu_1+\mu_2)^2 + \mu_2^2\lambda_4\lambda_5) + 3\lambda_4\lambda_6(\mu_1+\mu_2)(3\mu_1+\mu_2)(3\mu_1+2\mu_2)r^{4\mu_1} \right) \\
&\quad \left. + 3\mu_1\lambda_5\lambda_6(\mu_1+\mu_2)(3\mu_1+\mu_2)(3\mu_1+2\mu_2)r^{8\mu_1+6\mu_2} \right] \cos(2\mu_1+\mu_2)\theta, \\
\tilde{U}_{1,c53,1} &= \rho_1 \left[ \frac{\mu_1\lambda_2r^{\mu_1+\mu_2}}{2(3\mu_1+2\mu_2)} - \frac{\mu_1\lambda_3(2\mu_1+\mu_2)r^{3\mu_1+\mu_2}}{2\mu_2(3\mu_1+2\mu_2)} + \lambda_5r^{5\mu_1+3\mu_2} \right. \\
&\quad \left. - \frac{\lambda_6(2\mu_1+\mu_2)r^{7\mu_1+3\mu_2}}{2\mu_2} \right] \cos(\mu_1+\mu_2)\theta, \\
\tilde{U}_{2,c53,1} &= \frac{4\rho_2}{2\mu_2(3\mu_1+2\mu_2)} \left[ r^{3(\mu_1+\mu_2)} \left( 2r^{2\mu_1}(\mu_2(2\lambda_1\lambda_5(\mu_1+\mu_2)^2(3\mu_1+2\mu_2) + \mu_1^2\lambda_2\lambda_4(3\mu_1+\mu_2)) \right. \right. \\
&\quad \left. - \lambda_0\lambda_6(2\mu_1+\mu_2)^2(3\mu_1+\mu_2)(3\mu_1+2\mu_2)) + 2\mu_2(2\mu_1+\mu_2)(\lambda_0\lambda_5(3\mu_1+2\mu_2)^2 + \mu_1^2\lambda_2\lambda_3) \right. \\
&\quad \left. - (2\mu_1+\mu_2)^2r^{4\mu_1}(\lambda_1\lambda_6(3\mu_1+2\mu_2)^2 + \mu_1^2\lambda_3\lambda_4) \right) - \mu_1^2r^{\mu_1+\mu_2}(\lambda_0\lambda_3(2\mu_1+\mu_2)^2 + \lambda_1\mu_2^2\lambda_2) \\
&\quad - \mu_1r^{5(\mu_1+\mu_2)}(2(2\mu_1+\mu_2)r^{2\mu_1}(\lambda_2\lambda_6(3\mu_1+\mu_2)^2 + \lambda_3\lambda_5(\mu_1+\mu_2)^2) \\
&\quad - 3\mu_2\lambda_2\lambda_5(2\mu_1+\mu_2)(3\mu_1+2\mu_2) + 2(3\mu_1+2\mu_2)r^{4\mu_1}(\lambda_3\lambda_6(2\mu_1+\mu_2)^2 + \mu_2^2\lambda_4\lambda_5)) \\
&\quad \left. + 3\mu_1\mu_2\lambda_5\lambda_6(2\mu_1+\mu_2)(3\mu_1+2\mu_2)r^{11\mu_1+7\mu_2} \right] \cos(\mu_1+\mu_2)\theta, \\
\tilde{U}_{1,c54,1} &= \rho_1 \left[ \lambda_5r^{6\mu_1+3\mu_2} + \frac{\lambda_2r^{2\mu_1+\mu_2}\mu_1(3\mu_1+\mu_2)}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)} \right] \cos \mu_2\theta, \\
\tilde{U}_{2,c54,1} &= \frac{4\rho_2 r^{\mu_2}}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)} \left[ \lambda_0(\mu_1+\mu_2)(3\mu_1+\mu_2)(\mu_1^2\lambda_2 + \lambda_5(3\mu_1+2\mu_2)^2r^{2(2\mu_1+\mu_2)}) \right. \\
&\quad \left. + r^{2(2\mu_1+\mu_2)} \left( \lambda_5(\mu_1+\mu_2)^2(3\mu_1+2\mu_2)r^{2\mu_1}(2\lambda_1(2\mu_1+\mu_2) + \mu_1r^{2(\mu_1+\mu_2)}(\lambda_3 + \lambda_6r^{2(2\mu_1+\mu_2)})) \right. \right. \\
&\quad \left. + \mu_1\lambda_2(\mu_1(3\mu_1+\mu_2)(\lambda_3(\mu_1+\mu_2) + 2\lambda_4(2\mu_1+\mu_2)r^{2\mu_1}) \right. \\
&\quad \left. \left. + r^{2(\mu_1+\mu_2)}(6\lambda_5(\mu_1+\mu_2)(2\mu_1+\mu_2)^2 + \lambda_6(3\mu_1+\mu_2)^2(3\mu_1+2\mu_2)r^{2\mu_1})) \right) \right] \cos \mu_2\theta,
\end{aligned}$$

$$\begin{aligned}
\tilde{U}_{1,c_{61},1} &= \rho_1 \left[ \lambda_6 r^{5\mu_1+2\mu_2} + \frac{\lambda_5 r^{3\mu_1+2\mu_2} \mu_2 (\mu_1 + \mu_2)}{(2\mu_1 + \mu_2)(3\mu_1 + \mu_2)} \right] \cos(3\mu_1 + 2\mu_2)\theta, \\
\tilde{U}_{2,c_{61},1} &= \frac{4\rho_2 r^{3\mu_1+2\mu_2}}{(2\mu_1 + \mu_2)(3\mu_1 + \mu_2)} \left[ 2\mu_1 (\lambda_0 \lambda_6 (2\mu_1 + \mu_2)^2 (3\mu_1 + \mu_2) - \lambda_1 \mu_2 \lambda_5 (\mu_1 + \mu_2)^2) \right. \\
&\quad - r^{2\mu_2} \left( \mu_2 \lambda_2 (2\mu_1 + \mu_2) (\lambda_5 (\mu_1 + \mu_2)^2 + \lambda_6 (3\mu_1 + \mu_2)^2 r^{2\mu_1}) \right. \\
&\quad + \lambda_3 (\mu_1 + \mu_2) (2\mu_1 + \mu_2) r^{2\mu_1} (\mu_2 \lambda_5 (\mu_1 + \mu_2) + \lambda_6 (2\mu_1 + \mu_2) (3\mu_1 + \mu_2) r^{2\mu_1}) \\
&\quad + \lambda_4 (\mu_1 + \mu_2) (3\mu_1 + \mu_2) r^{4\mu_1} (\mu_2^2 \lambda_5 + \lambda_6 (2\mu_1 + \mu_2)^2 r^{2\mu_1}) \left. \right) \\
&\quad \left. - 6\mu_1^2 \lambda_5 \lambda_6 (\mu_1 + \mu_2) (2\mu_1 + \mu_2) r^{6\mu_1+4\mu_2} \right] \cos(3\mu_1 + 2\mu_2)\theta, \\
\tilde{U}_{1,c_{62},1} &= \rho_1 \left[ \lambda_6 r^{5\mu_1+3\mu_2} - \frac{\lambda_4 r^{3\mu_1+\mu_2} \mu_1 \mu_2}{(2\mu_1 + \mu_2)(3\mu_1 + 2\mu_2)} \right] \cos(3\mu_1 + \mu_2)\theta, \\
\tilde{U}_{2,c_{62},1} &= \frac{4\rho_2 r^{3\mu_1+\mu_2}}{(2\mu_1 + \mu_2)(3\mu_1 + 2\mu_2)} \left[ r^{2\mu_2} (2(\mu_1 + \mu_2) (\lambda_0 \lambda_6 (2\mu_1 + \mu_2)^2 (3\mu_1 + 2\mu_2) + \mu_1^2 \mu_2 \lambda_2 \lambda_4) \right. \\
&\quad + \mu_2 (2\mu_1 + \mu_2) r^{2\mu_1} (\lambda_1 \lambda_6 (3\mu_1 + 2\mu_2)^2 + \mu_1^2 \lambda_3 \lambda_4)) \\
&\quad + \mu_1^2 \lambda_1 \mu_2 \lambda_4 (2\mu_1 + \mu_2) - \mu_1 r^{4(\mu_1+\mu_2)} ((3\mu_1 + 2\mu_2) (\lambda_3 \lambda_6 (2\mu_1 + \mu_2)^2 + \mu_2^2 \lambda_4 \lambda_5) \\
&\quad \left. + 6\lambda_4 \lambda_6 (\mu_1 + \mu_2)^2 (2\mu_1 + \mu_2) r^{2\mu_1}) - \mu_1 \lambda_5 \lambda_6 (2\mu_1 + \mu_2)^2 (3\mu_1 + 2\mu_2) r^{6(\mu_1+\mu_2)} \right] \cos(3\mu_1 + \mu_2)\theta,
\end{aligned}$$

and by replacing the cos terms by sin terms, we have the  $\tilde{U}_{c_{ij},2}$ .

For simplicity of notations, we also denote by  $(Z_1, Z_2, \dots, Z_{14})$  the kernels  $(Z_{\lambda_0}, Z_{\lambda_1}, \dots, Z_{c_{62},2})$ . Because  $\{Z_i\}$  are linearly independent, we have

$$\det\left[\left(\int_{\mathbb{R}^2} \Delta Z_i \cdot Z_j\right)_{i,j=1,\dots,14}\right] \neq 0. \quad (2.16)$$

We have the following corollary:

**Corollary 2.1.** *If  $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$  satisfies  $|\phi(z)| \leq C(1 + |z|)^\alpha$  for some  $0 \leq \alpha < 1$ , and*

$$\begin{cases} \Delta \phi_1 + 2|z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \phi_1 - 3|z|^{2N_2} e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} \phi_2 = 0 \\ \Delta \phi_2 + 2|z|^{2N_2} e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} \phi_2 - |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \phi_1 = 0, \end{cases} \quad (2.17)$$

then  $\phi$  belongs to the following linear space  $\mathcal{K}^*$ : the span of

$$\{Z_{\lambda_4}^*, Z_{\lambda_5}^*, Z_{c_{43},1}^*, Z_{c_{43},2}^*, Z_{c_{52},1}^*, Z_{c_{52},2}^*, Z_{c_{53},1}^*, Z_{c_{53},2}^*, Z_{c_{54},1}^*, Z_{c_{54},2}^*, Z_{c_{61},1}^*, Z_{c_{61},2}^*, Z_{c_{62},1}^*, Z_{c_{62},2}^*\},$$

where

$$Z_i^* = \begin{pmatrix} Z_{i,1}^* \\ Z_{i,2}^* \end{pmatrix} = \begin{pmatrix} 2Z_{i,1} - Z_{i,2} \\ \frac{2}{3}Z_{i,2} - Z_{i,1} \end{pmatrix}. \quad (2.18)$$

We have

$$\det[(\int_{\mathbb{R}^2} Z_i^* \cdot Z_j^*)_{i,j=3,\dots,14}] \neq 0. \quad (2.19)$$

We will choose the first approximate solution to be  $\begin{pmatrix} \tilde{U}_{1,(\lambda, c_{43}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62})} \\ \tilde{U}_{2,(\lambda, c_{43}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62})} \end{pmatrix}$ , where the parameters  $\lambda, c_{43}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62}$  satisfy

$$|\mathbf{a}| := |c_{43}| + |c_{52}| + |c_{53}| + |c_{54}| + |c_{61}| + |c_{62}| \leq C_0 \varepsilon, \quad |\lambda| = O(1) \quad (2.20)$$

for some fixed constant  $C_0 > 0$ .

For the simplicity of notations, we also denote  $\mathbf{b} = (\lambda, \mathbf{a})$ , and  $\tilde{U}_{i,\mathbf{b}} = \tilde{U}_{i,(\lambda, c_{43}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62})}$ . We want to look for solutions of the form

$$\tilde{U}_1 = \tilde{U}_{1,\mathbf{b}} + \varepsilon \Psi_1 + \varepsilon^2 \phi_1, \quad \tilde{U}_2 = \tilde{U}_{2,\mathbf{b}} + \varepsilon \Psi_2 + \varepsilon^2 \phi_2, \quad (2.21)$$

where  $\mathbf{b}$  is fixed.

To obtain the next order term, we need to study the linearized operator around the solution  $\begin{pmatrix} \tilde{U}_{1,0} \\ \tilde{U}_{2,0} \end{pmatrix}$ .

### 2.3 Invertibility of the Linearized Operator

Now we consider the invertibility of the linearized operator in some suitable Sobolev spaces. To this end, we use the technical framework introduced by Chae-Imanuvilov [4] and which has been used in [1]. Let  $\alpha \in (0, 1)$  and

$$X_\alpha = \{u \in L_{loc}^2(\mathbb{R}^2), \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha}) u^2 dx < +\infty\}, \quad (2.22)$$

$$Y_\alpha = \{u \in W_{loc}^{2,2}(\mathbb{R}^2), \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha}) |\Delta u|^2 + \frac{u^2}{1 + |x|^{2+\alpha}} < +\infty\}. \quad (2.23)$$

On  $X_\alpha$  and  $Y_\alpha$ , we equip with two norms respectively:

$$\|f\|_{**} = \sup_{y \in \mathbb{R}^2} (1 + |y|)^{2+\alpha} |f(y)|, \quad \|h\|_* = \sup_{y \in \mathbb{R}^2} (\log(2 + |y|))^{-1} |h(y)|. \quad (2.24)$$

Clearly, the linearized operator in (2.15) is bounded from  $Y_\alpha$  to  $X_\alpha$ .

For  $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ ,  $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ , we denote by  $\langle f, g \rangle = \int_{\mathbb{R}^2} f \cdot g dx$ .

Using the non-degeneracy result we get, i.e. Lemma 2.1 and Corollary 2.1 in Section 2.2 and noting that (2.17) is the adjoint operator of (2.15), we have the following:

**Lemma 2.2.** *Assume that  $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in X_\alpha$  be such that*

$$\langle Z_i^*, h \rangle = 0, \text{ for } i = 3, \dots, 14. \quad (2.25)$$

Then one can find a unique solution  $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = T^{-1}(h) \in Y_\alpha$  satisfying

$$\begin{cases} \Delta\phi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\phi_1 - \phi_2) = h_1 \\ \Delta\phi_2 + |z|^{2N_1} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\phi_2 - 3\phi_1) = h_2 \end{cases}, \quad \|\phi\|_* \leq C\|h\|_{**}, \quad (2.26)$$

such that  $\langle \Delta Z_i, \phi \rangle = 0$  for  $i = 1, \dots, 14$ . Moreover, the map  $h \xrightarrow{T} \phi$  can be made continuous and smooth.

We note that the uniqueness in Lemma 2.2 is due to (2.16). In the next subsections, we will use Lemma 2.2 to obtain our approximate solution up to  $O(\varepsilon^2)$ . However our approximation solution would be chosen according to our assumption of the vortex configuration.

## 2.4 Improvements of the Approximate Solution

Similar to the  $\mathbf{A}_2$  and  $\mathbf{B}_2$  case, we need to find the  $O(\varepsilon)$  and  $O(\varepsilon^2)$  improvement of our approximate solutions. So we need to find solutions of the following equations.

Denote by

$$f(\varepsilon, z) = \Pi_1^{N_1} |z - \varepsilon p_i|^2, \quad g(\varepsilon, z) = \Pi_1^{N_2} |z - \varepsilon q_i|^2.$$

Then by Taylor's expansion, we have

$$f(\varepsilon, z) = f(0, z) + \varepsilon f_\varepsilon(0, z) + \frac{\varepsilon^2}{2} f_{\varepsilon\varepsilon}(0, z) + O(\varepsilon^3)$$

where

$$f(0, z) = |z|^{2N_1}, \quad f_\varepsilon(0, z) = -2|z|^{2N_1-2} < \sum_{j=1}^{N_1} p_j, z >. \quad (2.27)$$

Similarly we can get the expansions for  $g(\varepsilon, z)$ .

Let  $\begin{pmatrix} \Psi_{0,1} \\ \Psi_{0,2} \end{pmatrix}$  be the solution of

$$\begin{cases} \Delta\Psi_{0,1} + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{0,1} - \Psi_{0,2}) \\ \quad = -f_\varepsilon(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}}, \\ \Delta\Psi_{0,2} + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\Psi_{0,2} - 3\Psi_{0,1}) \\ \quad = -g_\varepsilon(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}}. \end{cases} \quad (2.28)$$

Let  $\begin{pmatrix} \Psi_{i,1} \\ \Psi_{i,2} \end{pmatrix}$  be the solution of

$$\begin{cases} \Delta\Psi_{i,1} + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{i,1} - \Psi_{i,2}) = -|z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{0,1} - \Psi_{0,2}) (2Z_{i,1} - Z_{i,2}) \\ \quad - f_\varepsilon(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{i,1} - Z_{i,2}) \\ \Delta\Psi_{i,2} + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\Psi_{i,2} - 3\Psi_{i,1}) = -|z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\Psi_{0,2} - 3\Psi_{0,1}) (2Z_{i,2} - 3Z_{i,1}) \\ \quad - g_\varepsilon(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{i,2} - 3Z_{i,1}) \end{cases} \quad (2.29)$$



for  $i = 3, \dots, 14$ .

Let  $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$  be the solution of

$$\begin{cases} \Delta\psi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1 - \psi_2) = 2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} - |z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} \\ -\frac{1}{2}|z|^{2N_1} e^{2U_{1,0} - U_{2,0}} (2\Psi_{0,1} - \Psi_{0,2})^2 - f_\varepsilon(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\Psi_{0,1} - \Psi_{0,2}) - \frac{f_{\varepsilon\varepsilon}(0, z)}{2} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} \\ \Delta\psi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2 - 3\psi_1) = 2|z|^{4N_2} e^{4\tilde{U}_{2,0} - 6\tilde{U}_{1,0}} - 3|z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} \\ -\frac{1}{2}|z|^{2N_2} e^{2U_{2,0} - 3U_{1,0}} (2\Psi_{0,2} - 3\Psi_{0,1})^2 - g_\varepsilon(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\Psi_{0,2} - 3\Psi_{0,1}) - \frac{g_{\varepsilon\varepsilon}(0, z)}{2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}}. \end{cases} \quad (2.30)$$

Obviously, if  $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$ , then  $f_\varepsilon(0, z) = g_\varepsilon(0, z) = 0$  by (2.27), in this case,  $\Psi_0 = \Psi_i = 0$  for  $i = 3, \dots, 14$ . Note that if  $N_2 \sum_{j=1}^{N_1} p_j = N_1 \sum_{j=1}^{N_2} q_j$ , we can always shift the origin such that  $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$ . Hence  $f_\varepsilon(0, z) \neq 0$  occurs only when assumption (b) of Theorem 1.1 holds. In this case, using Lemma 2.2, we know that there exists a unique solution  $\begin{pmatrix} \psi_{0,1} \\ \psi_{0,2} \end{pmatrix} \in Y_\alpha$  of (2.28) such that  $\langle \Psi_0, \Delta Z_j \rangle = 0$  for  $j = 1, \dots, 14$ . If  $N_2 \sum_{j=1}^{N_1} p_j \neq N_1 \sum_{j=1}^{N_2} q_j$ , then  $|N_1 - N_2| \neq 1$  implies that the right hand side of equation (2.29) is orthogonal to  $Z_i^*$  for  $i = 3, \dots, 14$ . By Lemma 2.2, there exists a unique solution  $\begin{pmatrix} \Psi_{i,1} \\ \Psi_{i,2} \end{pmatrix} \in Y_\alpha$  of (2.29) such that  $\langle \Psi_i, \Delta Z_j \rangle = 0$  for  $j = 1, \dots, 14$ . And if  $N_1, N_2 > 1$ , the right hand side of (2.30) is orthogonal to  $Z_i^*$  for  $i = 3, \dots, 14$ . By Lemma 2.2, we can find a unique solution  $\psi_0 = \begin{pmatrix} \psi_{0,1} \\ \psi_{0,2} \end{pmatrix} \in Y_\alpha$  such that  $\langle \psi_0, \Delta Z_i \rangle = 0$  for  $i = 1, \dots, 14$ .

Similar to the  $\mathbf{A}_2$  and  $\mathbf{B}_2$  case, the solution we will use later is  $\psi = \psi_0 + \xi_1 Z_{\lambda_4} + \xi_2 Z_{\lambda_5}$  where  $\xi_1, \xi_2$  are two constants independent of  $\mathbf{a}$  and will be determined later.

Finally, the approximate solution with all the terms of  $O(\varepsilon)$  and  $O(\varepsilon^2)$  is

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \tilde{U}_{1,\mathbf{b}} + \varepsilon(\Psi_{0,1} + \sum_{i=3}^{14} \Psi_{i,1} \mathbf{a}_i) + \varepsilon^2 \psi_1 \\ \tilde{U}_{2,\mathbf{b}} + \varepsilon(\Psi_{0,2} + \sum_{i=3}^{14} \Psi_{i,2} \mathbf{a}_i) + \varepsilon^2 \psi_2 \end{pmatrix}, \quad (2.31)$$

we use the notation

$$\begin{aligned} \mathbf{b} &= (\lambda_4, \lambda_5, \mathbf{a}), \\ \mathbf{a} &= (\mathbf{a}_3, \mathbf{a}_4, \dots, \mathbf{a}_{14}) \\ &= (c_{43,1}, c_{43,2}, c_{52,1}, c_{52,2}, c_{53,1}, c_{53,2}, c_{54,1}, c_{54,2}, c_{61,1}, c_{61,2}, c_{62,1}, c_{62,2}). \end{aligned}$$

Parameters  $\lambda_4, \lambda_5$  will be chosen later according to different assumptions in Theorem 1.1. After  $\lambda_4, \lambda_5$  are fixed,  $\mathbf{a}$  will be chosen in order to find a solution of (2.2) in the form of (2.31). Then  $\begin{pmatrix} V_1 + \varepsilon^2 v_1 \\ V_2 + \varepsilon^2 v_2 \end{pmatrix}$  is a solution of (2.2) if  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  satisfies

$$\begin{cases} \Delta v_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2v_1 - v_2) = G_1 \\ \Delta v_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2v_2 - 3v_1) = G_2, \end{cases} \quad (2.32)$$

where

$$G_1 = E_1 + N_{11}(v) + N_{12}(v), \quad (2.33)$$

$$G_2 = E_2 + N_{21}(v) + N_{22}(v), \quad (2.34)$$

$$\begin{aligned} N_{11}(v) &= 2\Pi|z - \varepsilon p_j|^4(e^{4\tilde{U}_1-2\tilde{U}_2} - e^{4V_1-2V_2}) \\ &\quad - \Pi|z - \varepsilon p_j|^2\Pi|z - \varepsilon q_j|^2(e^{\tilde{U}_2-\tilde{U}_1} - e^{V_2-V_1}), \\ N_{12}(v) &= \frac{-f(\varepsilon, z)e^{2\tilde{U}_1-\tilde{U}_2} + f(\varepsilon, z)e^{2V_1-V_2}}{\varepsilon^2} \\ &\quad + f(0, z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2v_1 - v_2), \\ N_{21}(v) &= 2\Pi|z - \varepsilon q_j|^4(e^{4\tilde{U}_2-6\tilde{U}_1} - e^{4V_2-6V_1}) \\ &\quad - 3\Pi|z - \varepsilon p_j|^2\Pi|z - \varepsilon q_j|^2(e^{\tilde{U}_2-\tilde{U}_1} - e^{V_2-V_1}), \\ N_{22}(v) &= \frac{-g(\varepsilon, z)e^{2\tilde{U}_2-3\tilde{U}_1} + g(\varepsilon, z)e^{2V_2-3V_1}}{\varepsilon^2} \\ &\quad + g(0, z)e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2v_2 - 3v_1), \end{aligned}$$

and  $E_i$  are the errors:

$$\begin{aligned} E_1 &= 2\Pi|z - \varepsilon p_j|^4e^{4V_1-2V_2} - \Pi|z - \varepsilon p_j|^2\Pi|z - \varepsilon q_j|^2e^{V_2-V_1} - 2|z|^{4N_1}e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}} \\ &\quad + |z|^{2N_1+2N_2}e^{\tilde{U}_{2,0}-\tilde{U}_{1,0}} + \frac{E_{11}}{\varepsilon^2} \\ &\quad + f(0, z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\psi_1 - \psi_2) \\ &\quad + \frac{f(0, z)}{2}e^{2U_{1,0}-U_{2,0}}(2\Psi_{0,1} - \Psi_{0,2})^2 + f_\varepsilon(0, z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\Psi_{0,1} - \Psi_{0,2}) \\ &\quad + \frac{f_{\varepsilon\varepsilon}(0, z)}{2}e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}, \end{aligned}$$

$$\begin{aligned} E_2 &= 2\Pi|z - \varepsilon q_j|^4e^{4V_2-6V_1} - 3\Pi|z - \varepsilon p_j|^2\Pi|z - \varepsilon q_j|^2e^{V_2-V_1} - 2|z|^{4N_2}e^{4\tilde{U}_{2,0}-6\tilde{U}_{1,0}} \\ &\quad + 3|z|^{2N_1+2N_2}e^{\tilde{U}_{2,0}-\tilde{U}_{1,0}} + \frac{E_{22}}{\varepsilon^2} \\ &\quad + g(0, z)e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2\psi_2 - 3\psi_1) \\ &\quad + \frac{g(0, z)}{2}e^{2U_{2,0}-3U_{1,0}}(2\Psi_{0,2} - 3\Psi_{0,1})^2 + g_\varepsilon(0, z)e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2\Psi_{0,2} - 3\Psi_{0,1}) \\ &\quad + \frac{g_{\varepsilon\varepsilon}(0, z)}{2}e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}. \end{aligned}$$

Here

$$\begin{aligned} E_{11} &= -f(\varepsilon, z)e^{2V_1-V_2} + f(0, z)e^{2\tilde{U}_{1,b}-\tilde{U}_{2,b}} + \varepsilon f(0, z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2\Psi_{0,1} - \Psi_{0,2}) + \varepsilon f_\varepsilon(0, z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \\ &\quad + \sum_{i=3}^{14} \varepsilon[f(0, z)e^{2U_{1,0}-U_{2,0}}(2\Psi_{i,1} - \Psi_{i,2}) + f(0, z)e^{2U_{1,0}-U_{2,0}}(2\Psi_{0,1} - \Psi_{0,2})(2Z_{i,1} - Z_{i,2})]\mathbf{a}_i \\ &\quad + \sum_{i=3}^{14} \varepsilon f_\varepsilon(0, z)e^{2U_{1,0}-U_{2,0}}(2Z_{i,1} - Z_{i,2})\mathbf{a}_i \end{aligned}$$

and

$$\begin{aligned}
E_{22} = & -g(\varepsilon, z)e^{2V_2-3V_1} + g(0, z)e^{2\tilde{U}_{2,\mathbf{b}}-3\tilde{U}_{1,\mathbf{b}}} + \varepsilon g(0, z)e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2\Psi_{0,2} - 3\Psi_{0,1}) \\
& + \varepsilon g_\varepsilon(0, z)e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} \\
& + \sum_{i=3}^{14} \varepsilon [g(0, z)e^{2U_{2,0}-3U_{1,0}}(2\Psi_{i,2} - 3\Psi_{i,1}) + g(0, z)e^{2U_{2,0}-3U_{1,0}}(2\Psi_{0,2} - 3\Psi_{0,1})(2Z_{i,2} - 3Z_{i,1})] \mathbf{a}_i \\
& + \sum_{i=3}^{14} \varepsilon g_\varepsilon(0, z)e^{2U_{2,0}-3U_{1,0}}(2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i.
\end{aligned}$$

By Taylor's expansion, we have

$$\begin{aligned}
E_1 = & -f(0, z)e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \left\{ \frac{1}{\varepsilon} \sum_{i=3}^{14} (2\Psi_{i,1} - \Psi_{i,2}) \mathbf{a}_i \sum_{j=3}^{14} (2Z_{j,1} - Z_{j,2}) \mathbf{a}_j \right. \\
& + (2\psi_1 - \psi_2) \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i + \frac{1}{2} (2\Psi_{0,1} - \Psi_{0,2})^2 \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \\
& + (2\Psi_{0,1} - \Psi_{0,2}) \sum_{i=3}^{14} (2\Psi_{i,1} - \Psi_{i,2}) \mathbf{a}_i + (2\Psi_{0,1} - \Psi_{0,2}) \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \Big\} \\
& - \frac{f(0, z)}{2\varepsilon} \sum_{i,j=3}^{14} \partial_{\mathbf{a}_i \mathbf{a}_j}^2 (e^{(2\tilde{U}_{1,0}-\tilde{U}_{2,0})}) (2\Psi_{0,1} - \Psi_{0,2}) \mathbf{a}_i \mathbf{a}_j \\
& - f_\varepsilon(0, z) \left\{ e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2\Psi_{0,1} - \Psi_{0,2}) \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i + e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \sum_{i=3}^{14} (2\Psi_{i,1} - \Psi_{i,2}) \mathbf{a}_i \right. \\
& + \frac{1}{2\varepsilon} \sum_{i,j=3}^{14} \partial_{\mathbf{a}_i \mathbf{a}_j}^2 (e^{(2\tilde{U}_{1,0}-\tilde{U}_{2,0})}) \mathbf{a}_i \mathbf{a}_j \Big\} - \frac{1}{2} f_{\varepsilon\varepsilon}(0, z) e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \\
& + 4f(0, z)^2 e^{4\tilde{U}_{1,0}-2\tilde{U}_{2,0}} \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i - f(0, z)g(0, z) \sum_{i=3}^{14} e^{\tilde{U}_{2,0}-\tilde{U}_{1,0}} (Z_{i,2} - Z_{i,1}) \mathbf{a}_i \\
& + O(\varepsilon) + O(\varepsilon^2 + |\mathbf{a}|^2),
\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}
E_2 = & -g(0, z)e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} \left\{ \frac{1}{\varepsilon} \sum_{i=3}^{14} (2\Psi_{i,2} - 3\Psi_{i,1}) \mathbf{a}_i \sum_{j=3}^{14} (2Z_{j,2} - 3Z_{j,1}) \mathbf{a}_j \right. \\
& + (2\psi_2 - 3\psi_1) \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i + \frac{1}{2} (2\Psi_{0,2} - 3\Psi_{0,1})^2 \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i \\
& + (2\Psi_{0,2} - 3\Psi_{0,1}) \sum_{i=3}^{14} (2\Psi_{i,2} - 3\Psi_{i,1}) \mathbf{a}_i + (2\Psi_{0,2} - 3\Psi_{0,1}) \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i \Big\} \\
& - \frac{g(0, z)}{2\varepsilon} \sum_{i,j=3}^{14} \partial_{\mathbf{a}_i \mathbf{a}_j}^2 (e^{(2\tilde{U}_{2,0}-3\tilde{U}_{1,0})}) (2\Psi_{0,2} - 3\Psi_{0,1}) \mathbf{a}_i \mathbf{a}_j \\
& - g_\varepsilon(0, z) \left\{ e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} (2\Psi_{0,2} - 3\Psi_{0,1}) \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i + e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} \sum_{i=3}^{14} (2\Psi_{i,2} - 3\Psi_{i,1}) \mathbf{a}_i \right. \\
& + \frac{1}{2\varepsilon} \sum_{i,j=3}^{14} \partial_{\mathbf{a}_i \mathbf{a}_j}^2 (e^{(2\tilde{U}_{2,0}-3\tilde{U}_{1,0})}) \mathbf{a}_i \mathbf{a}_j \Big\} - \frac{1}{2} g_{\varepsilon\varepsilon}(0, z) e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i \\
& + 4g(0, z)^2 e^{4\tilde{U}_{2,0}-6\tilde{U}_{1,0}} \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i - 3f(0, z)g(0, z) \sum_{i=3}^{14} e^{\tilde{U}_{2,0}-\tilde{U}_{1,0}} (Z_{i,2} - Z_{i,1}) \mathbf{a}_i \\
& + O(\varepsilon) + O(\varepsilon^2 + |\mathbf{a}|^2),
\end{aligned} \tag{2.36}$$

where  $O(\varepsilon)$  denotes all items only involving with  $\varepsilon$ , and not with  $\mathbf{a}$ .

### 3 A Nonlinear Projected Problem

Similar to Proposition 2.1 in [1], we have the following result:

**Proposition 3.1.** *For  $\mathbf{a}$  satisfying (2.20), there exists a solution  $(v, \{m_i\})$  to the following system*

$$\begin{cases} \Delta v_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2v_1 - v_2) = G_1 + \sum_{i=3}^{14} m_i(v) Z_{i,1}^* \\ \Delta v_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} (2v_2 - 3v_1) = G_2 + \sum_{i=3}^{14} m_i(v) Z_{i,2}^* \\ \langle \Delta Z_i, v \rangle = 0, \text{ for } i = 1, \dots, 14, \end{cases} \tag{3.1}$$

where  $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$  and  $m_i(v)$  can be determined by

$$\langle G + \sum_{i=3}^{14} m_i(v) Z_i^*, Z_j^* \rangle = 0, \text{ for } j = 3, \dots, 14. \tag{3.2}$$

Furthermore,  $v$  satisfies the following estimate

$$\|v\|_* \leq C\varepsilon, \tag{3.3}$$

for some constant  $C$  independent of  $\varepsilon$ .

By Proposition 3.1, the full solvability for (2.2) is reduced to  $m_i = 0$  for  $i = 3, \dots, 14$ . Since by (2.19),  $\det(\langle Z_i^*, Z_j^* \rangle_{i,j=3,\dots,14}) \neq 0$ , and recall the definition of  $m_i$  in (3.2),  $m_i = 0$  is equivalent to

$$\int_0^{+\infty} \int_0^{2\pi} G \cdot Z_i^* r d\theta dr = 0 \quad \text{for } i = 3, \dots, 14. \quad (3.4)$$

To solve (3.4), we observe all  $N_{ij}$  terms in (2.33) and (2.34) are small. In fact, we have the following lemma:

**Lemma 3.1.** *Let  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  be a solution of (3.1). Then we have the following estimates:*

$$\int_{\mathbb{R}^2} (N_{11}(v) + N_{12}(v)) Z_{i,1}^* + (N_{21}(v) + N_{22}(v)) Z_{i,2}^* dx = O(\varepsilon^2), \quad (3.5)$$

for  $i = 3, \dots, 14$ , where  $O(\varepsilon^2) \leq C_1 \varepsilon^2$  for some positive  $C_1$  independent of  $\mathbf{a}$  provided  $|\mathbf{a}| \leq 1$ .

**Proof:**

The proof is similar to that of Lemma 2.4 in [1]. □

From the Taylor expansions in (2.35) and (2.36), we obtain that the error projection can be expressed as

$$(\langle E, Z_{\mathbf{a}_i}^* \rangle)_{i=3,\dots,14} = \frac{1}{\varepsilon} \tilde{\mathbf{A}} \mathbf{a} \cdot \mathbf{a} + \tilde{\mathbf{Q}} \mathbf{a} + O(|\mathbf{a}|^2) + O(\varepsilon). \quad (3.6)$$

## 4 Proof of Theorem 1.1 under Assumption (i)

Suppose the **Assumption (i)** holds. By a translation, we might assume that  $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j = 0$  and  $N_1 = N_2$ , and we choose  $(\xi_1, \xi_2) = (0, 0)$  in this section. This case is the reminiscent of  $SU(2)$  case, even though, the proof is considerably harder since there are fourteen dimensional kernels instead of a three-dimensional one for the  $SU(2)$  case.

**Lemma 4.1.** *Let  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  be a solution of (3.1). The following estimates hold:*

$$\begin{aligned} & (\langle E, Z_{c_{43,1}}^* \rangle, \langle E, Z_{c_{43,2}}^* \rangle, \langle E, Z_{c_{52,1}}^* \rangle, \langle E, Z_{c_{52,2}}^* \rangle, \\ & \langle E, Z_{c_{53,1}}^* \rangle, \langle E, Z_{c_{53,2}}^* \rangle, \langle E, Z_{c_{54,1}}^* \rangle, \langle E, Z_{c_{54,2}}^* \rangle, \\ & \langle E, Z_{c_{61,1}}^* \rangle, \langle E, Z_{c_{61,2}}^* \rangle, \langle E, Z_{c_{62,1}}^* \rangle, \langle E, Z_{c_{62,2}}^* \rangle)^t \\ & = \tilde{\mathcal{T}}(\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon), \end{aligned} \quad (4.1)$$

where  $\tilde{\mathcal{T}}$  is an  $12 \times 12$  matrix defined in (4.11). Moreover,  $\tilde{\mathcal{T}}$  is non-degenerate if  $N > 0$ .

**Proof:**

Without loss of generality, we may assume that  $\sum_{j=1}^{N_1} p_j = \sum_{j=1}^{N_2} q_j = 0$  and  $N_1 = N_2 = N$ , and denote by  $\mu = N + 1$ . Now we choose the parameters  $(\lambda_4, \lambda_5, \xi_1, \xi_2) = (\frac{1}{3 \times 2^{10} \mu^6}, \frac{1}{15 \times 2^9 \mu^6}, 0, 0)$ , so that we have

$$e^{\tilde{U}_{1,0}} = \rho_{1,G} = \frac{1}{2} \rho_1 = \frac{45 \times 2^9 \mu^6}{(1 + r^{2\mu})^6}, \quad e^{\tilde{U}_{2,0}} = \rho_{2,G} = \frac{1}{4} \rho_2 = \frac{3^3 \times 5^2 \times 2^{15} \mu^{10}}{(1 + r^{2\mu})^{10}}. \quad (4.2)$$

Since  $\sum_{i=1}^N p_i = \sum_{i=1}^N q_i = 0$ , we have  $f_\varepsilon(0, z) = 0$ ,  $\Psi_{0,1} = \Psi_{0,2} = 0$ . By (2.29), we have  $\Psi_{i,1} = \Psi_{i,2} = 0$ ,  $i = 3, \dots, 14$ . Recall that

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad (4.3)$$

where by (2.35), and (2.36), we have

$$\begin{aligned} E_1 &= -|z|^{2N} \frac{\rho_{1,G}^2}{\rho_{2,G}} (2\psi_1 - \psi_2) \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i - \frac{1}{2} f_{\varepsilon\varepsilon}(0, z) \frac{\rho_{1,G}^2}{\rho_{2,G}} \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \\ &\quad + 4|z|^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} \sum_{i=3}^{14} (2Z_{i,1} - Z_{i,2}) \mathbf{a}_i - |z|^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \sum_{i=3}^{14} (Z_{i,2} - Z_{i,1}) \mathbf{a}_i \\ &\quad + O(\varepsilon) + O(\varepsilon^2 + |\mathbf{a}|^2), \\ E_2 &= -|z|^{2N} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2\psi_2 - 3\psi_1) \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i - \frac{1}{2} g_{\varepsilon\varepsilon}(0, z) \frac{\rho_{2,G}^2}{\rho_{1,G}^3} \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i \\ &\quad + 4|z|^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} \sum_{i=3}^{14} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i - 3|z|^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \sum_{i=3}^{14} (Z_{i,2} - Z_{i,1}) \mathbf{a}_i \\ &\quad + O(\varepsilon) + O(\varepsilon^2 + |\mathbf{a}|^2), \end{aligned}$$

where  $f_{\varepsilon\varepsilon}(0, z)$  is

$$\begin{aligned} f_{\varepsilon\varepsilon}(0, z) &= 2|z|^{2(N-1)} \left( \sum_i |p_i|^2 + 2 \sum_{i \neq j} (p_{i1} \cos \theta + p_{i2} \sin \theta)(p_{j1} \cos \theta + p_{j2} \sin \theta) \right) \\ &= 2|z|^{2(N-1)} \left( \sum_i |p_i|^2 + \sum_{i \neq j} (p_{i1} p_{j1} - p_{i2} p_{j2}) \cos 2\theta + (p_{i1} p_{j2} + p_{i2} p_{j1}) \sin 2\theta \right) \\ &= 2|z|^{2(N-1)} \left( \sum_{i \neq j} (p_{i1} p_{j1} - p_{i2} p_{j2}) \cos 2\theta + (p_{i1} p_{j2} + p_{i2} p_{j1}) \sin 2\theta \right). \end{aligned} \quad (4.4)$$

Similarly we can get the expression for  $g_{\varepsilon\varepsilon}(0, z)$ .

Since

$$\begin{aligned} &\int h(r) \cos 2\theta (2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta \\ &= \int h(r) \sin 2\theta (2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta = 0, \end{aligned}$$

and

$$\begin{aligned} &\int h(r) \cos 2\theta (Z_{i,2} - Z_{i,1})(Z_{j,2} - Z_{j,1}) r dr d\theta \\ &= \int h(r) \sin 2\theta (Z_{i,2} - Z_{i,1})(Z_{j,2} - Z_{j,1}) r dr d\theta = 0, \end{aligned}$$

for  $i, j = 3, \dots, 14$ , from (4.4), we have

$$\int_0^\infty \int_0^{2\pi} f_{\varepsilon\varepsilon}(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta = 0, \quad (4.5)$$

and

$$\int_0^\infty \int_0^{2\pi} g_{\varepsilon\varepsilon}(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{i,2} - 3Z_{i,1})(2Z_{j,2} - 3Z_{j,1}) r dr d\theta = 0, \quad (4.6)$$

for  $i, j = 3, \dots, 14$ . Note that  $f_{\varepsilon\varepsilon}(0, z) = g_{\varepsilon\varepsilon}(0, z) = 0$  if  $N = 1$ .

Another important observation is the following:

$$\int_0^\infty \int_0^{2\pi} r^{2N} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1 - \psi_2)(2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta \quad (4.7)$$

$$= \int_0^\infty \int_0^{2\pi} f(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1^0 - \psi_2^0)(2Z_{i,1} - Z_{i,2})(2Z_{j,1} - Z_{j,2}) r dr d\theta,$$

$$\int_0^\infty \int_0^{2\pi} r^{2N} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2 - 3\psi_1)(2Z_{i,2} - 3Z_{i,1})(2Z_{j,2} - 3Z_{j,1}) r dr d\theta \quad (4.8)$$

$$= \int_0^\infty \int_0^{2\pi} g(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2^0 - 3\psi_1^0)(2Z_{i,2} - 3Z_{i,1})(2Z_{j,2} - 3Z_{j,1}) r dr d\theta,$$

where  $(\psi_1^0, \psi_2^0)$  is the radial solution of the following system:

$$\begin{cases} \Delta\psi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1 - \psi_2) = 2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} - |z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} \\ \Delta\psi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2 - 3\psi_1) = 2|z|^{4N_2} e^{4\tilde{U}_{2,0} - 6\tilde{U}_{1,0}} - 3|z|^{2(N_1+N_2)} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}}. \end{cases} \quad (4.9)$$

Obviously,  $(\psi_1^0, \psi_2^0)$  is the radial part of  $(\psi_{0,1}, \psi_{0,2})$ . Because of this observation, when dealing with the  $O(\varepsilon^2)$  approximation, we only need to consider the radial part of the solutions.

In fact we can choose  $\psi_1^0 = \psi$ ,  $\psi_2^0 = \frac{5}{3}\psi$  such that  $\psi$  is the solution of the following ODE:

$$\Delta\psi + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2} \psi = r^{4N} \frac{3 \times 2^6 (N+1)^4}{(1+r^{2N+2})^4}. \quad (4.10)$$

Combining (4.5), (4.6), (4.7) and (4.8), one can get the following:

$$\begin{aligned} & \int E \cdot Z_k^* r dr d\theta \\ &= \int_0^\infty \int_0^{2\pi} \sum_{i=3}^{14} \left( \left[ 2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} (4Z_{i,1} - 2Z_{i,2}) \mathbf{a}_i - |z|^{2N_1+2N_2} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} (Z_{i,2} - Z_{i,1}) \mathbf{a}_i \right. \right. \\ & - f(0, z) e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_{0,1} - \psi_{0,2})(2Z_{i,1} - Z_{i,2}) \mathbf{a}_i \left. \right] Z_{k,1}^* \\ & + \left[ 4|z|^{4N_2} e^{4\tilde{U}_{2,0} - 6\tilde{U}_{1,0}} (2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i - 3|z|^{2N_1+2N_2} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} (Z_{i,2} - Z_{i,1}) \mathbf{a}_i \right. \\ & - g(0, z) e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_{0,2} - 3\psi_{0,1})(2Z_{i,2} - 3Z_{i,1}) \mathbf{a}_i \left. \right] Z_{k,2}^* \Big) r dr d\theta \\ & + O(|\mathbf{a}|^2 + |\varepsilon|^2) + O(\varepsilon), \end{aligned}$$

where  $O(\varepsilon)$  is independent of  $\mathbf{a}$ .

Thus we can get that for  $i = 1, 2$

$$\begin{aligned}
& \int E \cdot Z_{c_{43,i}}^* r dr d\theta \\
&= \int \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} \left[ (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})^2 c_{43,i} + (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}}) c_{54,i} \right] \right. \\
&\quad - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{43,i,2}} - Z_{c_{43,i,1}})(2Z_{c_{43,i,1}} - Z_{c_{43,i,2}}) c_{43,i} + (Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{43,i,1}} - Z_{c_{43,i,2}}) c_{54,i} \right] \\
&\quad - r^{2N} \frac{\rho_{1,G}^2}{\rho_{2,G}} (2\psi_{1,0} - \psi_{2,0}) \left[ (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})^2 c_{43,i} + (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}}) c_{54,i} \right] \\
&\quad + 4r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} \left[ \frac{1}{3} (2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})^2 c_{43,i} + \frac{1}{3} (2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) c_{54,i} \right] \\
&\quad - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{43,i,2}} - Z_{c_{43,i,1}})(2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}}) c_{43,i} + (Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}}) c_{54,i} \right] \\
&\quad - r^{2N} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2\psi_{2,0} - 3\psi_{1,0}) \left[ \frac{1}{3} (2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})^2 c_{43,i} \right. \\
&\quad \left. + \frac{1}{3} (2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) c_{54,i} \right] \Big) r dr d\theta \\
&= \int \left[ \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{43,i,2}} - Z_{c_{43,i,1}})(2Z_{c_{43,i,1}} - Z_{c_{43,i,2}}) \right. \right. \\
&\quad + \frac{4}{3} r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} (2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{43,i,2}} - Z_{c_{43,i,1}})(2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}}) \Big) \\
&\quad - \frac{1}{3} r^{2N} \left( \frac{\rho_{1,G}^2}{\rho_{2,G}} (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})^2 + \frac{1}{3} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})^2 \right) \psi \Big) c_{43,i} \\
&\quad + \left( \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}}) - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{43,i,1}} - Z_{c_{43,i,2}}) \right. \right. \\
&\quad + \frac{4r^{4N}}{3} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} (2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) - \frac{\rho_{2,G} r^{4N}}{\rho_{1,G}} (Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}}) \Big) \\
&\quad - \frac{r^{2N}}{3} \left( \frac{\rho_{1,G}^2}{\rho_{2,G}} (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}}) \right. \\
&\quad \left. + \frac{1}{3} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) \right) \psi \Big) c_{54,i} \Big] r dr d\theta \\
&= \pi (J_1 + \int q_1 \psi r dr) c_{43,i} + (J_2 + \int q_7 \psi r dr) c_{54,i} \\
&+ O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}$$



similarly, we can get that

$$\begin{aligned}
& \int E \cdot Z_{c_{54,i}}^* r dr d\theta \\
&= \int \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} \left[ (2Z_{c_{54,i,1}} - Z_{c_{54,i,2}})^2 c_{54,i} + (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}}) c_{43,i} \right] \right. \\
&\quad - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}}) c_{54,i} + (Z_{c_{43,i,2}} - Z_{c_{43,i,1}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}}) c_{43,i} \right] \\
&\quad - r^{2N} \frac{\rho_{1,G}^2}{\rho_{2,G}} (2\psi_{1,0} - \psi_{2,0}) \left[ (2Z_{c_{54,i,1}} - Z_{c_{54,i,2}})^2 c_{54,i} + (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}}) c_{43,i} \right] \\
&\quad + 4r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} \left[ \frac{1}{3} (2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}})^2 c_{54,i} + \frac{1}{3} (2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}})(2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}}) c_{43,i} \right] \\
&\quad - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) c_{54,i} + (Z_{c_{43,i,2}} - Z_{c_{43,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) c_{43,i} \right] \\
&\quad - r^{2N} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2\psi_{2,0} - 3\psi_{1,0}) \left[ \frac{1}{3} (2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}})^2 c_{54,i} \right. \\
&\quad \left. + \frac{1}{3} (2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) c_{43,i} \right] \Big) r dr d\theta \\
&= \int \left[ \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} (2Z_{c_{54,i,1}} - Z_{c_{54,i,2}})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}}) \right. \right. \\
&\quad + \frac{4}{3} r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} (2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) \Big) \\
&\quad - \frac{1}{3} r^{2N} \left( \frac{\rho_{1,G}^2}{\rho_{2,G}} (2Z_{c_{54,i,1}} - Z_{c_{54,i,2}})^2 + \frac{1}{3} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}})^2 \right) \psi \Big) c_{54,i} \\
&\quad + \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} (2Z_{c_{43,i,1}} - Z_{c_{43,i,2}})(2Z_{c_{54,i,1}} - Z_{c_{54,i,2}}) - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{54,i,2}} - Z_{c_{54,i,1}})(2Z_{c_{43,i,1}} - Z_{c_{43,i,2}}) \right. \\
&\quad + \frac{4r^{4N}}{3} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} (2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) - \frac{\rho_{2,G} r^{4N}}{\rho_{1,G}} (Z_{c_{43,i,2}} - Z_{c_{43,i,1}})(2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}}) \\
&\quad - \frac{1}{3} r^{2N} \left( \frac{\rho_{1,G}^2}{\rho_{2,G}} (2Z_{c_{54,i,1}} - Z_{c_{54,i,2}})(2Z_{c_{43,i,1}} - Z_{c_{43,i,2}}) \right. \\
&\quad \left. + \frac{1}{3} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2Z_{c_{54,i,2}} - 3Z_{c_{54,i,1}})(2Z_{c_{43,i,2}} - 3Z_{c_{43,i,1}}) \right) \psi \Big) c_{43,i} \Big] r dr d\theta \\
&= \pi (J_3 + \int q_4 \psi r dr) c_{43,i} + (J_4 + \int q_7 \psi r dr) c_{54,i} \\
&\quad + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}$$

and

$$\begin{aligned}
& \int E \cdot Z_{c_{52},i}^* r dr d\theta \\
&= \int \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} \left[ (2Z_{c_{52},i,1} - Z_{c_{52},i,2})^2 c_{52,i} \right] - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{52},i,2} - Z_{c_{52},i,1})(2Z_{c_{52},i,1} - Z_{c_{52},i,2}) c_{52,i} \right] \right. \\
&\quad - r^{2N} \frac{\rho_{1,G}^2}{\rho_{2,G}} (2\psi_{1,0} - \psi_{2,0}) \left[ (2Z_{c_{52},i,1} - Z_{c_{52},i,2})^2 c_{52,i} \right] \\
&\quad + 4r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} \left[ \frac{1}{3} (2Z_{c_{52},i,2} - 3Z_{c_{52},i,1})^2 c_{52,i} \right] - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{52},i,2} - Z_{c_{52},i,1})(2Z_{c_{52},i,2} - 3Z_{c_{52},i,1}) c_{52,i} \right] \\
&\quad \left. - r^{2N} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2\psi_{2,0} - 3\psi_{1,0}) \left[ \frac{1}{3} (2Z_{c_{52},i,2} - 3Z_{c_{52},i,1})^2 c_{52,i} \right] \right) r dr d\theta \\
&= \int \left[ \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} (2Z_{c_{52},i,1} - Z_{c_{52},i,2})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{52},i,2} - Z_{c_{52},i,1})(2Z_{c_{52},i,1} - Z_{c_{52},i,2}) \right. \right. \\
&\quad + \frac{4}{3} r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} (2Z_{c_{52},i,2} - 3Z_{c_{52},i,1})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{52},i,2} - Z_{c_{52},i,1})(2Z_{c_{52},i,2} - 3Z_{c_{52},i,1}) \\
&\quad \left. \left. - \frac{1}{3} r^{2N} \left( \frac{\rho_{1,G}^2}{\rho_{2,G}} (2Z_{c_{52},i,1} - Z_{c_{52},i,2})^2 + \frac{1}{3} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2Z_{c_{52},i,2} - 3Z_{c_{52},i,1})^2 \right) \psi \right) c_{52,i} \right] r dr d\theta \\
&= \pi(J_5 + \int q_2 \psi r dr) c_{52,i} + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}$$

$$\begin{aligned}
& \int E \cdot Z_{c_{53},i}^* r dr d\theta \\
&= \int \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} \left[ (2Z_{c_{53},i,1} - Z_{c_{53},i,2})^2 c_{53,i} \right] - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{53},i,2} - Z_{c_{53},i,1})(2Z_{c_{53},i,1} - Z_{c_{53},i,2}) c_{53,i} \right] \right. \\
&\quad - r^{2N} \frac{\rho_{1,G}^2}{\rho_{2,G}} (2\psi_{1,0} - \psi_{2,0}) \left[ (2Z_{c_{53},i,1} - Z_{c_{53},i,2})^2 c_{53,i} \right] \\
&\quad + 4r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} \left[ \frac{1}{3} (2Z_{c_{53},i,2} - 3Z_{c_{53},i,1})^2 c_{53,i} \right] - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{53},i,2} - Z_{c_{53},i,1})(2Z_{c_{53},i,2} - 3Z_{c_{53},i,1}) c_{53,i} \right] \\
&\quad \left. - r^{2N} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2\psi_{2,0} - 3\psi_{1,0}) \left[ \frac{1}{3} (2Z_{c_{53},i,2} - 3Z_{c_{53},i,1})^2 c_{53,i} \right] \right) r dr d\theta \\
&= \int \left[ \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} (2Z_{c_{53},i,1} - Z_{c_{53},i,2})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{53},i,2} - Z_{c_{53},i,1})(2Z_{c_{53},i,1} - Z_{c_{53},i,2}) \right. \right. \\
&\quad + \frac{4}{3} r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} (2Z_{c_{53},i,2} - 3Z_{c_{53},i,1})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{53},i,2} - Z_{c_{53},i,1})(2Z_{c_{53},i,2} - 3Z_{c_{53},i,1}) \\
&\quad \left. \left. - \frac{1}{3} r^{2N} \left( \frac{\rho_{1,G}^2}{\rho_{2,G}} (2Z_{c_{53},i,1} - Z_{c_{53},i,2})^2 + \frac{1}{3} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2Z_{c_{53},i,2} - 3Z_{c_{53},i,1})^2 \right) \psi \right) c_{53,i} \right] r dr d\theta \\
&= \pi(J_6 + \int q_3 \psi r dr) c_{53,i} + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}$$

$$\begin{aligned}
& \int E \cdot Z_{c_{61,i}}^* r dr d\theta \\
&= \int \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} \left[ (2Z_{c_{61,i,1}} - Z_{c_{61,i,2}})^2 c_{61,i} \right] - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{61,i,2}} - Z_{c_{61,i,1}})(2Z_{c_{61,i,1}} - Z_{c_{61,i,2}}) c_{61,i} \right] \right. \\
&\quad - r^{2N} \frac{\rho_{1,G}^2}{\rho_{2,G}} (2\psi_{1,0} - \psi_{2,0}) \left[ (2Z_{c_{61,i,1}} - Z_{c_{61,i,2}})^2 c_{61,i} \right] \\
&\quad + 4r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} \left[ \frac{1}{3} (2Z_{c_{61,i,2}} - 3Z_{c_{61,i,1}})^2 c_{61,i} \right] - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{61,i,2}} - Z_{c_{61,i,1}})(2Z_{c_{61,i,2}} - 3Z_{c_{61,i,1}}) c_{61,i} \right] \\
&\quad \left. - r^{2N} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2\psi_{2,0} - 3\psi_{1,0}) \left[ \frac{1}{3} (2Z_{c_{61,i,2}} - 3Z_{c_{61,i,1}})^2 c_{61,i} \right] \right) r dr d\theta \\
&= \int \left[ \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} (2Z_{c_{61,i,1}} - Z_{c_{61,i,2}})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{61,i,2}} - Z_{c_{61,i,1}})(2Z_{c_{61,i,1}} - Z_{c_{61,i,2}}) \right. \right. \\
&\quad + \frac{4}{3} r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} (2Z_{c_{61,i,2}} - 3Z_{c_{61,i,1}})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{61,i,2}} - Z_{c_{61,i,1}})(2Z_{c_{61,i,2}} - 3Z_{c_{61,i,1}}) \Big) \\
&\quad \left. - \frac{1}{3} r^{2N} \left( \frac{\rho_{1,G}^2}{\rho_{2,G}} (2Z_{c_{61,i,1}} - Z_{c_{61,i,2}})^2 + \frac{1}{3} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2Z_{c_{61,i,2}} - 3Z_{c_{61,i,1}})^2 \right) \psi \right) c_{61,i} \\
&= \pi(J_7 + \int q_5 \psi r dr) c_{61,i} + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}$$

$$\begin{aligned}
& \int E \cdot Z_{c_{62,i}}^* r dr d\theta \\
&= \int \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} \left[ (2Z_{c_{62,i,1}} - Z_{c_{62,i,2}})^2 c_{62,i} \right] - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{62,i,2}} - Z_{c_{62,i,1}})(2Z_{c_{62,i,1}} - Z_{c_{62,i,2}}) c_{62,i} \right] \right. \\
&\quad - r^{2N} \frac{\rho_{1,G}^2}{\rho_{2,G}} (2\psi_{1,0} - \psi_{2,0}) \left[ (2Z_{c_{62,i,1}} - Z_{c_{62,i,2}})^2 c_{62,i} \right] \\
&\quad + 4r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} \left[ \frac{1}{3} (2Z_{c_{62,i,2}} - 3Z_{c_{62,i,1}})^2 c_{62,i} \right] - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} \left[ (Z_{c_{62,i,2}} - Z_{c_{62,i,1}})(2Z_{c_{62,i,2}} - 3Z_{c_{62,i,1}}) c_{62,i} \right] \\
&\quad \left. - r^{2N} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2\psi_{2,0} - 3\psi_{1,0}) \left[ \frac{1}{3} (2Z_{c_{62,i,2}} - 3Z_{c_{62,i,1}})^2 c_{62,i} \right] \right) r dr d\theta \\
&= \int \left[ \left( 4r^{4N} \frac{\rho_{1,G}^4}{\rho_{2,G}^2} (2Z_{c_{62,i,1}} - Z_{c_{62,i,2}})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{62,i,2}} - Z_{c_{62,i,1}})(2Z_{c_{62,i,1}} - Z_{c_{62,i,2}}) \right. \right. \\
&\quad + \frac{4}{3} r^{4N} \frac{\rho_{2,G}^4}{\rho_{1,G}^6} (2Z_{c_{62,i,2}} - 3Z_{c_{62,i,1}})^2 - r^{4N} \frac{\rho_{2,G}}{\rho_{1,G}} (Z_{c_{62,i,2}} - Z_{c_{62,i,1}})(2Z_{c_{62,i,2}} - 3Z_{c_{62,i,1}}) \Big) \\
&\quad \left. - \frac{1}{3} r^{2N} \left( \frac{\rho_{1,G}^2}{\rho_{2,G}} (2Z_{c_{62,i,1}} - Z_{c_{62,i,2}})^2 + \frac{1}{3} \frac{\rho_{2,G}^2}{\rho_{1,G}^3} (2Z_{c_{62,i,2}} - 3Z_{c_{62,i,1}})^2 \right) \psi \right) c_{62,i} \\
&= \pi(J_8 + \int q_6 \psi r dr) c_{62,i} + O(\varepsilon) + O(|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}$$

where all the terms  $O(\varepsilon) \leq C\varepsilon$ ,  $O(\varepsilon^2) \leq C\varepsilon^2$ ,  $O(|\mathbf{a}|^2) \leq C|\mathbf{a}|^2$  for some positive constant  $C$  independent of  $\mathbf{a}$  and  $\varepsilon$  provided that they are small enough.

So we get that

$$\begin{aligned}
& (\langle E, Z_{c_{43,1}}^* \rangle, \langle E, Z_{c_{43,2}}^* \rangle, \langle E, Z_{c_{52,1}}^* \rangle, \langle E, Z_{c_{52,2}}^* \rangle, \\
& \quad \langle E, Z_{c_{53,1}}^* \rangle, \langle E, Z_{c_{53,2}}^* \rangle, \langle E, Z_{c_{54,1}}^* \rangle, \langle E, Z_{c_{54,2}}^* \rangle, \\
& \quad \langle E, Z_{c_{61,1}}^* \rangle, \langle E, Z_{c_{61,2}}^* \rangle, \langle E, Z_{c_{62,1}}^* \rangle, \langle E, Z_{c_{62,2}}^* \rangle)^t \\
& = \tilde{\mathcal{T}}(\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon),
\end{aligned}$$

and

$$\tilde{\mathcal{T}} = \begin{pmatrix} T_1 & 0 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_1 & 0 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ T_5 & 0 & 0 & 0 & 0 & 0 & T_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_5 & 0 & 0 & 0 & 0 & 0 & T_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_8 \end{pmatrix}, \quad (4.11)$$

where

$$\begin{aligned}
T_1 &= (J_1 + \int q_1 \psi r dr), \quad T_2 = (J_2 + \int q_7 \psi r dr), \\
T_3 &= (J_5 + \int q_2 \psi r dr), \quad T_4 = (J_6 + \int q_3 \psi r dr), \\
T_5 &= (J_3 + \int q_4 \psi r dr), \quad T_6 = (J_4 + \int q_7 \psi r dr), \\
T_7 &= (J_7 + \int q_5 \psi r dr), \quad T_8 = (J_8 + \int q_6 \psi r dr)
\end{aligned}$$

The determinant of the matrix  $\tilde{\mathcal{T}}$  is

$$\begin{aligned}
& (T_1 T_6 - T_2 T_5)^2 T_3^3 T_4^2 T_7^2 T_8^2 \\
& = (J_5 + \int q_2 \psi r dr)^2 (J_6 + \int q_3 \psi r dr)^2 (J_7 + \int q_5 \psi r dr)^2 (J_8 + \int q_6 \psi r dr)^2 \\
& \times [(J_1 + \int q_1 \psi r dr)(J_4 + \int q_7 \psi r dr) - (J_2 + \int q_7 \psi r dr)(J_3 + \int q_4 \psi r dr)]^2
\end{aligned}$$

Next we prove that the matrix  $\tilde{\mathcal{T}}$  is non-degenerate, i.e. the determinant of  $\tilde{\mathcal{T}}$  is nonzero. For this purpose, we need to calculate the integrals  $J_1$  to  $J_8$ , and  $\int q_1 \psi r dr$  to  $\int q_7 \psi r dr$ . But in the integrals, there is the function  $\psi$  for which the expression is unknown. In order to get rid of  $\psi$ , we use integration by parts. The key observation is that for any  $\phi$  satisfying  $\phi(\infty) = 0$ , we have

$$\int_0^\infty \left[ \left( \Delta + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2} \right) \psi \right] \phi r dr = \int_0^\infty \left[ \left( \Delta + \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2} \right) \phi \right] \psi r dr. \quad (4.12)$$

By direct calculation, one can get that

$$\begin{aligned}
q_1 &= -\frac{1}{3}r^{2N}[e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{c_{43},1}(r) - Z_{c_{43},2}(r))^2 + \frac{1}{3}e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2Z_{c_{43},2}(r) - 3Z_{c_{43},1}(r))^2] \\
&= -18\frac{\mu^2r^{2\mu+2N}}{(r^{2\mu}+1)^{12}}(-178r^{2\mu} + 1252r^{4\mu} - 3746r^{6\mu} + 5380r^{8\mu} - 3746r^{10\mu} + 1252r^{12\mu} \\
&\quad -178r^{14\mu} + 9r^{16\mu} + 9), \\
q_2 &= -\frac{1}{3}r^{2N}[e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{c_{52},1}(r) - Z_{c_{52},2}(r))^2 + \frac{1}{3}e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2Z_{c_{52},2}(r) - 3Z_{c_{52},1}(r))^2] \\
&= -2240\frac{\mu^2r^{6\mu+2N}}{(r^{2\mu}+1)^{12}}(-5r^{2\mu} + 2r^{4\mu} + 2)^2, \\
q_3 &= -\frac{1}{3}r^{2N}[e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{c_{53},1}(r) - Z_{c_{53},2}(r))^2 + \frac{1}{3}e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2Z_{c_{53},2}(r) - 3Z_{c_{53},1}(r))^2] \\
&= -1260\frac{\mu^2r^{4\mu+2N}}{(r^{2\mu}+1)^{12}}(5r^{2\mu} - 5r^{4\mu} + r^{6\mu} - 1)^2, \\
q_4 &= -\frac{1}{3}r^{2N}[e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{c_{54},1}(r) - Z_{c_{54},2}(r))^2 + \frac{1}{3}e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2Z_{c_{54},2}(r) - 3Z_{c_{54},1}(r))^2] \\
&= -32\frac{\mu^2r^{2\mu+2N}}{9(r^{2\mu}+1)^{12}}\left(-578r^{2\mu} + 4052r^{4\mu} \right. \\
&\quad \left. -12146r^{6\mu} + 17420r^{8\mu} - 12146r^{10\mu} + 4052r^{12\mu} - 578r^{14\mu} + 29r^{16\mu} + 29\right), \\
q_5 &= -\frac{1}{3}r^{2N}[e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{c_{61},1}(r) - Z_{c_{61},2}(r))^2 + \frac{1}{3}e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2Z_{c_{61},2}(r) - 3Z_{c_{61},1}(r))^2] \\
&= \frac{-560\mu^2r^{10\mu+2N}}{(r^{2\mu}+1)^{12}}, \\
q_6 &= -\frac{1}{3}r^{2N}[e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{c_{62},1}(r) - Z_{c_{62},2}(r))^2 + \frac{1}{3}e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2Z_{c_{62},2}(r) - 3Z_{c_{62},1}(r))^2] \\
&= \frac{-560\mu^2r^{8\mu+2N}(r^{2\mu}-1)^2}{(r^{2\mu}+1)^{12}}, \\
q_7 &= -\frac{1}{3}r^{2N}[e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}}(2Z_{c_{54},1}(r) - Z_{c_{54},2}(r))(2Z_{c_{43},1}(r) - Z_{c_{43},2}(r)) \\
&\quad + \frac{1}{3}e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}}(2Z_{c_{54},2}(r) - 3Z_{c_{54},1}(r))(2Z_{c_{43},2}(r) - 3Z_{c_{43},1}(r))] \\
&= 16\frac{\mu^2r^{2\mu+2N}}{(r^{2\mu}+1)^{12}}(-11r^{2\mu} + 21r^{4\mu} - 11r^{6\mu} + r^{8\mu} + 1)(-73r^{2\mu} + 153r^{4\mu} - 73r^{6\mu} + 8r^{8\mu} + 8),
\end{aligned}$$

where we denote by  $Z_{c_{ij},k}(r)$  the radial part of  $Z_{c_{ij},k}$ .

So we need to find solutions of ODEs:

$$\Delta\phi_i + \frac{8(N+1)^2r^{2N}}{(1+r^{2N+2})^2}\phi_i = q_i, \quad (4.13)$$

for  $q_i$  defined as above and  $i = 1, \dots, 7$ .

Following the same idea as in the proof of Lemma 3.4 in [1], one can get that

$$\begin{aligned}
\phi_1 &= -\frac{(81r^{16\mu} - 352r^{14\mu} + 1393r^{12\mu} - 1584r^{10\mu} + 1435r^{8\mu} - 304r^{6\mu} + 108r^{4\mu} + 8r^{2\mu} + 1)}{8(r^{2\mu} + 1)^{10}}, \\
\phi_2 &= -\frac{2(350r^{12\mu} - 336r^{10\mu} + 392r^{8\mu} + 48r^{6\mu} + 27r^{4\mu} + 8r^{2\mu} + 1)}{5(r^{2\mu} + 1)^{10}}, \\
\phi_3 &= -\frac{(350r^{14\mu} - 875r^{12\mu} + 1764r^{10\mu} - 833r^{8\mu} + 398r^{6\mu} + 27r^{4\mu} + 8r^{2\mu} + 1)}{10(r^{2\mu} + 1)^{10}}, \\
\phi_4 &= -\frac{2(145r^{16\mu} - 640r^{14\mu} + 2485r^{12\mu} - 2896r^{10\mu} + 2527r^{8\mu} - 592r^{6\mu} + 172r^{4\mu} + 8r^{2\mu} + 1)}{45(r^{2\mu} + 1)^{10}}, \\
\phi_5 &= -\frac{5(42r^{8\mu} + 48r^{6\mu} + 27r^{4\mu} + 8r^{2\mu} + 1)}{54(r^{2\mu} + 1)^{10}}, \\
\phi_6 &= -\frac{4(189r^{10\mu} + 42r^{8\mu} + 48r^{6\mu} + 27r^{4\mu} + 8r^{2\mu} + 1)}{135(r^{2\mu} + 1)^{10}}, \\
\phi_7 &= \frac{r^{4\mu}(16r^{12\mu} - 72r^{10\mu} + 273r^{8\mu} - 328r^{6\mu} + 273r^{4\mu} - 72r^{2\mu} + 16)}{2(r^{2\mu} + 1)^{10}}.
\end{aligned}$$

So by direct calculation, we have

$$\int \psi q_i r dr = \int_0^\infty r^{4N} \frac{3 \times 2^6 (N+1)^4}{(1+r^{2N+2})^4} \phi_i r dr.$$

Since all the terms in the integrals are explicit now, by direct calculation, we get

$$\begin{aligned}
&J_5 + \int q_2 \psi r dr \\
&= \frac{\pi N \csc(\frac{\pi}{N+1})}{5405400(N+1)^{10}} \\
&\quad \times \left( 229219200N^{11} + 2741981760N^{10} + 14845935288N^9 + 48087228720N^8 \right. \\
&\quad \left. + 103607697806N^7 + 155921208688N^6 + 167142190971N^5 + 127469747650N^4 \right. \\
&\quad \left. + 67655345084N^3 + 23742751992N^2 + 4943660256N + 461099520 \right), \\
&J_6 + \int q_3 \psi r dr \\
&= \frac{N \pi \csc(\frac{\pi}{N+1})}{5405400(N+1)^{10}} \\
&\quad \times \left( 73483200N^{11} + 879636240N^{10} + 4780746072N^9 + 15595933680N^8 + 33964635664N^7 \right. \\
&\quad \left. + 51871736672N^6 + 56687585199N^5 + 44305756250N^4 + 24240925096N^3 \right. \\
&\quad \left. + 8824079448N^2 + 1917285264N + 187548480 \right),
\end{aligned}$$

$$\begin{aligned}
& J_7 + \int q_5 \psi r dr \\
&= \frac{\pi N \csc(\frac{\pi}{N+1})}{70053984(N+1)^{10}} \\
&\quad \times \left( 604195200N^{11} + 7224396480N^{10} + 38670644088N^9 + 122313524400N^8 \right. \\
&\quad \left. + 253958797454N^7 + 363332551792N^6 + 365322970803N^5 + 257996555026N^4 \right. \\
&\quad \left. + 125308378700N^3 + 39819491064N^2 + 7439155488N + 617621760 \right), \\
& J_8 + \int q_6 \psi r dr \\
&= \frac{\pi N \csc(\frac{\pi}{N+1})}{437837400(N+1)^{10}} \\
&\quad \times \left( (N+2)(2N+3)(3N+4)(4N+5)(5N+6)(N(N(2N(12N(504N(855N+3997) \right. \right. \\
&\quad \left. \left. + 4024843) + 51916217) + 62504971) + 19787638) + 2554776) \right), \\
& (J_1 + \int q_1 \psi r dr)(J_4 + \int q_7 \psi r dr) - (J_2 + \int q_7 \psi r dr)(J_3 + \int q_4 \psi r dr) \\
&= \frac{\pi^2 N^3 \csc^2(\frac{\pi}{N+1})}{6949800(N+1)^{12}} \left( (N+2)^2(2N+3)^2(N(N(N(N(2N(9N(2N(120N(4365N+37031) \right. \right. \\
&\quad \left. \left. + 17041651) + 77384503) + 1031119715) + 2064834729) + 1398543708) \right. \\
&\quad \left. + 617719460) + 161120544) + 18780480) \right).
\end{aligned}$$

One can check that the above terms are all nonzero if  $N > 0$ . So  $\tilde{\mathcal{T}}$  is non-degenerate if  $N > 0$ . □

As a consequence of Lemma 3.1 and Lemma 4.1, the coefficients  $m_i$  vanish if and only if the parameters  $\mathbf{a}$  satisfy

$$\tilde{\mathcal{T}}(\mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon) = 0, \quad (4.14)$$

where  $\tilde{\mathcal{T}}$  is defined in (4.11). Obviously, (4.14) can be solved immediately from (4.11) with  $|\mathbf{a}| \leq C\varepsilon$ , for some  $C$  large but fixed.

If  $N = 0$ , we choose the zero-th approximate solutions to be radial  $\tilde{U}_0$ , and the final approximate solution to be radial, then one can easily find a radial solution  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  to (3.1) with  $m_i$  all zero in Proposition 3.1, since  $G$  is radial, (3.4) is automatically satisfied.

## 5 Proof of Theorem 1.1 under Assumption (ii)

In this section, we are going to prove Theorem 1.1 under **Assumption (ii)**. This situation is more complicated than the previous one, since the  $O(\varepsilon)$  approximation and  $O(\varepsilon^2)$  approximation induce several difficulties. The problem is that we cannot obtain the

explicit expressions for these terms. In this case, we will see that the two free parameters  $\xi_1, \xi_2$  we introduced in Section 2.4 for the improvement of the  $O(\varepsilon^2)$  approximate solution play an important role. A key observation is that we only need to consider the terms involving  $\xi_1$  and  $\xi_2$ . This is contained in the following lemma.

**Lemma 5.1.** *Let  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  be a solution of (3.1). The following estimates hold:*

$$\langle E, Z_{c_{43,i}}^* \rangle \quad (5.1)$$

$$= \xi_1(\tilde{\mathcal{A}}_1 c_{43,i} + \tilde{\mathcal{B}}_1 c_{54,i}) + \xi_2(\tilde{\mathcal{A}}_2 c_{43,i} + \tilde{\mathcal{B}}_2 c_{54,i}) + \tilde{\mathcal{T}}_{1i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

$$\langle E, Z_{c_{52,i}}^* \rangle \quad (5.2)$$

$$= \xi_1 \tilde{\mathcal{C}}_1 c_{52,i} + \xi_2 \tilde{\mathcal{C}}_2 c_{52,i} + \tilde{\mathcal{T}}_{3i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

$$\langle E, Z_{c_{53,i}}^* \rangle \quad (5.3)$$

$$= \xi_1 \tilde{\mathcal{D}}_1 c_{53,i} + \xi_2 \tilde{\mathcal{D}}_2 c_{53,i} + \tilde{\mathcal{T}}_{4i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

$$\langle E, Z_{c_{54,i}}^* \rangle \quad (5.4)$$

$$= \xi_1(\tilde{\mathcal{E}}_1 c_{54,i} + \tilde{\mathcal{F}}_1 c_{43,i}) + \xi_2(\tilde{\mathcal{E}}_2 c_{54,i} + \tilde{\mathcal{F}}_2 c_{43,i}) + \tilde{\mathcal{T}}_{2i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

$$\langle E, Z_{c_{61,i}}^* \rangle \quad (5.5)$$

$$= \xi_1 \tilde{\mathcal{G}}_1 c_{61,i} + \xi_2 \tilde{\mathcal{G}}_2 c_{61,i} + \tilde{\mathcal{T}}_{4i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

$$\langle E, Z_{c_{62,i}}^* \rangle \quad (5.6)$$

$$= \xi_1 \tilde{\mathcal{H}}_1 c_{62,i} + \xi_2 \tilde{\mathcal{H}}_2 c_{62,i} + \tilde{\mathcal{T}}_{4i}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O(\varepsilon),$$

for  $i = 1, 2$ , where

$$\begin{aligned} \tilde{\mathcal{A}}_j &= \int_0^{+\infty} \int_0^{2\pi} \left[ r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2}) (2Z_{c_{43,1},1} - Z_{c_{43,1},2})^2 \right. \\ &\quad \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) (2Z_{c_{43,1},2} - 3Z_{c_{43,1},1})^2 \right] r dr d\theta, \end{aligned} \quad (5.7)$$

$$\begin{aligned} \tilde{\mathcal{B}}_j &= \int_0^{+\infty} \int_0^{2\pi} \left[ r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2}) (2Z_{c_{43,1},1} - Z_{c_{43,1},2}) (2Z_{c_{54,1},1} - Z_{c_{54,1},2}) \right. \\ &\quad \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) (2Z_{c_{54,1},2} - 3Z_{c_{54,1},1}) (2Z_{c_{43,1},2} - 3Z_{c_{43,1},1}) \right] r dr d\theta, \end{aligned} \quad (5.8)$$

$$\begin{aligned} \tilde{\mathcal{C}}_j &= \int_0^{+\infty} \int_0^{2\pi} \left[ r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2}) (2Z_{c_{52,1},1} - Z_{c_{52,1},2})^2 \right. \\ &\quad \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) (2Z_{c_{52,1},2} - 3Z_{c_{52,1},1})^2 \right] r dr d\theta, \end{aligned} \quad (5.9)$$

$$\begin{aligned} \tilde{\mathcal{D}}_j &= \int_0^{+\infty} \int_0^{2\pi} \left[ r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2}) (2Z_{c_{53,1},1} - Z_{c_{53,1},2})^2 \right. \\ &\quad \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) (2Z_{c_{53,1},2} - 3Z_{c_{53,1},1})^2 \right] r dr d\theta, \end{aligned} \quad (5.10)$$



$$\begin{aligned}\tilde{\mathcal{E}}_j &= \int_0^{+\infty} \int_0^{2\pi} \left[ r^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2}) (2Z_{c_{54,1,1}} - Z_{c_{54,1,2}})^2 \right. \\ &\quad \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) (2Z_{c_{54,1,2}} - 3Z_{c_{54,1,1}})^2 \right] r dr d\theta, \quad (5.11)\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{F}}_j &= \int_0^{+\infty} \int_0^{2\pi} \left[ r^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2}) (2Z_{c_{43,1,1}} - Z_{c_{43,1,2}}) (2Z_{c_{54,1,1}} - Z_{c_{54,1,2}}) \right. \\ &\quad \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) (2Z_{c_{54,1,2}} - 3Z_{c_{54,1,1}}) (2Z_{c_{43,1,2}} - 3Z_{c_{43,1,1}}) \right] r dr d\theta, \quad (5.12)\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{G}}_j &= \int_0^{+\infty} \int_0^{2\pi} \left[ r^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2}) (2Z_{c_{61,1,1}} - Z_{c_{61,1,2}})^2 \right. \\ &\quad \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) (2Z_{c_{61,1,2}} - 3Z_{c_{61,1,1}})^2 \right] r dr d\theta, \quad (5.13)\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{H}}_j &= \int_0^{+\infty} \int_0^{2\pi} \left[ r^{2N_1} e^{2\tilde{U}_{1,0}-\tilde{U}_{2,0}} (2Z_{j,1} - Z_{j,2}) (2Z_{c_{62,1,1}} - Z_{c_{62,1,2}})^2 \right. \\ &\quad \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0}-3\tilde{U}_{1,0}} (2Z_{j,2} - 3Z_{j,1}) (2Z_{c_{62,1,2}} - 3Z_{c_{62,1,1}})^2 \right] r dr d\theta, \quad (5.14)\end{aligned}$$

for  $j = 1, 2$ , and  $\tilde{\mathcal{T}}_{ij}$  are  $12 \times 1$  vectors which are uniformly bounded as  $\varepsilon$  tends to 0, and are independent of  $\xi_1, \xi_2$ .

**Proof:**

By (2.35) and (2.36),  $E$  is of the form

$$\frac{1}{\varepsilon} (\dots) \mathbf{a} \cdot \mathbf{a} + ((\dots) \mathbf{a}) + O(|\mathbf{a}|^2) + O(\varepsilon). \quad (5.15)$$

Recall that  $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_{0,1} \\ \psi_{0,2} \end{pmatrix} + \xi_1 Z_{\lambda_4} + \xi_2 Z_{\lambda_5}$ . In the following computations, we only need to consider the terms involving  $\xi_1$  and  $\xi_2$ , since all other terms are independent

of  $\xi_1$  and  $\xi_2$ . By the orthogonality of  $\cos(k\theta)$  and  $\cos(l\theta)$  for  $k \neq l$ , we obtain

$$\begin{aligned}
& - \int_0^{+\infty} \int_0^{2\pi} E \cdot Z_{c_{43,1}}^* r d\theta dr \\
& = \int_0^\infty \int_0^{2\pi} \left[ r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} \left( \xi_1 (2Z_{1,1} - Z_{1,2}) + \xi_2 (2Z_{2,1} - Z_{2,2}) \right) \right. \\
& \times (2Z_{c_{43,1,1}} - Z_{c_{43,1,2}}) c_{43,1} Z_{c_{43,1,1}}^* \\
& + r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} \left( \xi_1 (2Z_{1,2} - 3Z_{1,1}) + \xi_2 (2Z_{2,2} - 3Z_{2,1}) \right) \\
& \times (2Z_{c_{43,1,2}} - 3Z_{c_{43,1,1}}) c_{43,1} Z_{c_{43,1,2}}^* \\
& + r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} \left( \xi_1 (2Z_{1,1} - Z_{1,2}) + \xi_2 (2Z_{2,1} - Z_{2,2}) \right) \\
& \times (2Z_{c_{54,1,1}} - Z_{c_{54,1,2}}) c_{54,1} Z_{c_{43,1,1}}^* \\
& + r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} \left( \xi_1 (2Z_{1,2} - 3Z_{1,1}) + \xi_2 (2Z_{2,2} - 3Z_{2,1}) \right) \\
& \times (2Z_{c_{54,1,2}} - 3Z_{c_{54,1,1}}) c_{54,1} Z_{c_{43,1,2}}^* \left. \right] r dr d\theta \\
& + \tilde{\mathcal{T}}_{11}(\mathbf{a}) + O\left(\frac{|\mathbf{a}|^2}{\varepsilon}\right) + O(\varepsilon) + O((1 + |\xi|)|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}$$

where  $\tilde{\mathcal{T}}_{11}(\mathbf{a})$  is the remaining terms which is a linear combinations of  $\mathbf{a}$  which comes from the remaining terms of  $O(\varepsilon^2)$  of  $E$ , and the coefficients of the linear combinations are uniformly bounded and are independent of  $\xi_1, \xi_2, \mathbf{a}$ . The  $O(\frac{|\mathbf{a}|^2}{\varepsilon})$  terms comes from the  $O(\frac{|\mathbf{a}|^2}{\varepsilon})$  term of  $E$  which is independent of  $\xi$ .

Thus

$$\begin{aligned}
& - \int_0^{+\infty} \int_0^{2\pi} E \cdot Z_{c_{43,1}}^* r d\theta dr \\
& = \xi_1 \left[ \left( \int_0^{+\infty} \int_0^{2\pi} \left\{ r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{1,1} - Z_{1,2}) (2Z_{c_{43,1,1}} - Z_{c_{43,1,2}})^2 \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{1,2} - 3Z_{1,1}) (2Z_{c_{43,1,2}} - 3Z_{c_{43,1,1}})^2 \right\} r dr d\theta \right) c_{43,1} \right. \\
& \quad \left. + \left( \int_0^{+\infty} \int_0^{2\pi} \left\{ r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{1,1} - Z_{1,2}) (2Z_{c_{54,1,1}} - Z_{c_{54,1,2}}) (2Z_{c_{43,1,1}} - Z_{c_{43,1,2}}) \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{1,2} - 3Z_{1,1}) (2Z_{c_{43,1,2}} - 3Z_{c_{43,1,1}}) (2Z_{c_{54,1,2}} - 3Z_{c_{54,1,1}}) \right\} r dr d\theta \right) c_{54,1} \right] \\
& \quad + \xi_2 \left[ \left( \int_0^{+\infty} \int_0^{2\pi} \left\{ r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{2,1} - Z_{2,2}) (2Z_{c_{43,1,1}} - Z_{c_{43,1,2}})^2 \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{2,2} - 3Z_{2,1}) (Z_{c_{43,1,2}} - Z_{c_{43,1,1}})^2 \right\} r dr d\theta \right) c_{43,1} \right. \\
& \quad \left. + \left( \int_0^{+\infty} \int_0^{2\pi} \left\{ r^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2Z_{2,1} - Z_{2,2}) (2Z_{c_{43,1,1}} - Z_{c_{43,1,2}}) (2Z_{c_{54,1,1}} - Z_{c_{54,1,2}}) \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{1}{3} r^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2Z_{2,2} - 3Z_{2,1}) (2Z_{c_{43,1,2}} - 3Z_{c_{43,1,1}}) (2Z_{c_{54,1,2}} - 3Z_{c_{54,1,1}}) \right\} r dr d\theta \right) c_{54,1} \right] \\
& \quad + \tilde{\mathcal{T}}_{11}(\mathbf{a}) + O\left(\frac{|\mathbf{a}|^2}{\varepsilon}\right) + O(\varepsilon) + O((1 + |\xi|)|\mathbf{a}|^2 + \varepsilon^2) \\
& = \xi_1(\tilde{\mathcal{A}}_1 c_{43,1} + \tilde{\mathcal{B}}_1 c_{54,1}) + \xi_2(\tilde{\mathcal{A}}_2 c_{43,1} + \tilde{\mathcal{B}}_2 c_{54,1}) \\
& \quad + \tilde{\mathcal{T}}_{11}(\mathbf{a}) + O\left(\frac{|\mathbf{a}|^2}{\varepsilon}\right) + O(\varepsilon) + O((1 + |\xi|)|\mathbf{a}|^2 + \varepsilon^2),
\end{aligned}$$

where  $\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}_2, \tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2$  are in (5.7) and (5.8).

Similarly, we can get the other estimates. □

From the above lemma, we have the following result:

**Lemma 5.2.** *Let  $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  be a solution of (3.1). Then the coefficients  $m_i = 0$  if and only if the parameters  $\mathbf{a}$  satisfy*

$$\tilde{\mathbf{Q}}(\mathbf{a}) = \tilde{\mathcal{T}}(\mathbf{a}) + O\left(\frac{|\mathbf{a}|^2}{\varepsilon}\right) + O((1 + |\xi|)|\mathbf{a}|^2) + O(\varepsilon), \tag{5.16}$$

where  $\tilde{\mathbf{Q}}$

$$\tilde{\mathbf{Q}} = \xi_1 \begin{pmatrix} \tilde{\mathcal{A}}_1 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{B}}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\mathcal{A}}_1 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{B}}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\mathcal{C}}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{C}}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\mathcal{D}}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{D}}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{\mathcal{F}}_1 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{E}}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\mathcal{F}}_1 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{E}}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{G}}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{G}}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{H}}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{H}}_1 \end{pmatrix} \quad (5.17)$$

$$+ \xi_2 \begin{pmatrix} \tilde{\mathcal{A}}_2 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{B}}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\mathcal{A}}_2 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{B}}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\mathcal{C}}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{C}}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\mathcal{D}}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{D}}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{\mathcal{F}}_2 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{E}}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\mathcal{F}}_2 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{E}}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{G}}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{G}}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{H}}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\mathcal{H}}_2 \end{pmatrix} \quad (5.18)$$

$$= \xi_1 \mathbf{Q}_1 + \xi_2 \mathbf{Q}_2,$$

and  $\tilde{\mathcal{T}}$  is a  $12 \times 12$  matrix which is uniformly bounded and independent of  $\xi_1, \xi_2$ .

**Proof of Theorem 1.1 under Assumption (ii):** Under the **Assumptions (ii)**, we will choose  $\lambda_4 = \frac{\tilde{\lambda}}{(2^{7/2}\mu_1\mu_2(\mu_1+\mu_2))^2}$  and  $\lambda_5 = \frac{1}{\tilde{\lambda}}$  for  $\tilde{\lambda}$  large enough. Similar to the computation in the appendix of [1], by direct but tedious computation, we can get that for  $\tilde{\lambda}$  large,

$$\begin{aligned} \tilde{\mathcal{A}}_1 &= \gamma_1 \tilde{\lambda}^{-1} + o(\tilde{\lambda}^{-1}), \\ \tilde{\mathcal{C}}_1 &= \gamma_2 \tilde{\lambda}^{-1} + o(\tilde{\lambda}^{-1}), \\ \tilde{\mathcal{D}}_1 &= \gamma_3 \tilde{\lambda}^{-2} + o(\tilde{\lambda}^{-2}), \\ \tilde{\mathcal{E}}_1 &= \gamma_4 \tilde{\lambda}^{-4} + o(\tilde{\lambda}^{-4}), \\ \tilde{\mathcal{G}}_1 &= \gamma_5 \tilde{\lambda}^{-3} + o(\tilde{\lambda}^{-3}), \\ \tilde{\mathcal{H}}_1 &= \gamma_6 + o(1), \end{aligned}$$

and

$$\tilde{\mathcal{B}}_1 = \tilde{\mathcal{F}}_1 = \begin{cases} O(\tilde{\lambda}^{-3}) & \text{if } N_1 = N_2, \\ 0 & \text{if } N_1 \neq N_2, \end{cases} \quad (5.19)$$

where  $\gamma_1$  to  $\gamma_6$  are non zero as follows:

$$\begin{aligned}
\gamma_1 &= \frac{4092\mu_1(\mu_1 + \mu_2)(3\mu_1 + \mu_2)}{2\mu_1 + \mu_2} \pi^2 \csc\left(\frac{\mu_1\pi}{2\mu_1 + \mu_2}\right), \\
\gamma_2 &= -44695552\mu_1^4\mu_2^2(\mu_1 + \mu_2)^2(2\mu_1 + \mu_2)^3(3\mu_1 + \mu_2)^2\pi, \\
\gamma_3 &= \mu_1^3\mu_2^2 2^{36 - \frac{56\mu_1}{2\mu_1 + \mu_2}} (2\mu_1 - \mu_2)(6\mu_1 - \mu_2)(\mu_1 + \mu_2)^4(2\mu_1 + \mu_2)(3\mu_1 + 2\mu_2) \\
&\quad \times \left( (\mu_1 + \mu_2)^2(2\mu_1 + \mu_2)^2(3\mu_1 + 2\mu_2)^2 \right)^{1 - \frac{8\mu_1}{2\mu_1 + \mu_2}} (2\mu_1^2 - 5\mu_1\mu_2 - 6\mu_2^2)\pi^2 \csc\left(\frac{8\mu_1\pi}{2\mu_1 + \mu_2}\right), \\
\gamma_4 &= \frac{\mu_1^4\mu_2^5 2^{\frac{14\mu_1}{2\mu_1 + \mu_2} + 17}}{(3\mu_1 + \mu_2)^2} (\mu_1 + \mu_2)^6(2\mu_1 + \mu_2)((\mu_1 + \mu_2)^2(2\mu_1 + \mu_2)^2(3\mu_1 + 2\mu_2)^2)^{-\frac{2(\mu_1 + \mu_2)}{2\mu_1 + \mu_2}} \\
&\quad \times (9\mu_1 + 7\mu_2)\pi^2 \csc\left(\frac{2\mu_1\pi}{2\mu_1 + \mu_2}\right), \\
\gamma_5 &= -341\mu_1^6\mu_2^6 2^{\frac{7\mu_1}{2\mu_1 + \mu_2} + 23} (\mu_1 + \mu_2)^9(3\mu_1 + 2\mu_2)^3 \\
&\quad \times ((\mu_1 + \mu_2)^2(2\mu_1 + \mu_2)^2(3\mu_1 + 2\mu_2)^2)^{\frac{\mu_1}{2\mu_1 + \mu_2} - 1} \times (5\mu_1 + 3\mu_2)\pi^2 \csc\left(\frac{\pi\mu_1}{2\mu_1 + \mu_2}\right), \\
\gamma_6 &= -\frac{1}{3}\mu_1^4\mu_2^2 2^{\frac{7\mu_1}{2\mu_1 + \mu_2} + 15} (\mu_1 + \mu_2)^5(2\mu_1 + \mu_2)(3\mu_1 + \mu_2)(3\mu_1 + 2\mu_2)^2 \\
&\quad \times ((\mu_1 + \mu_2)(2\mu_1 + \mu_2)(3\mu_1 + 2\mu_2))^{\frac{2\mu_1}{2\mu_1 + \mu_2} - 4} \pi^2 \csc\left(\frac{\pi\mu_1}{2\mu_1 + \mu_2}\right).
\end{aligned}$$

It is easy to see that from the above expressions that  $\gamma_1, \gamma_2, \gamma_4, \gamma_5, \gamma_6$  are all non zero, and  $\gamma_3$  is also non zero, since for  $\mu_2 = 2\mu_1$  or  $\mu_2 = 6\mu_1$ ,  $\gamma_3$  is also non zero. So we have

$$\begin{aligned}
\tilde{\mathcal{A}}_1 \tilde{\mathcal{E}}_1 - \tilde{\mathcal{B}}_1 \tilde{\mathcal{F}}_1 &= \gamma_1 \gamma_4 \tilde{\lambda}^{-5} + o(\tilde{\lambda}^{-5}) \\
&\neq 0,
\end{aligned}$$

and  $\tilde{\mathcal{C}}_1, \tilde{\mathcal{D}}_1, \tilde{\mathcal{E}}_1, \tilde{\mathcal{G}}_1, \tilde{\mathcal{H}}_1$  are both non-zero if  $\tilde{\lambda}$  is large enough. Therefore, we choose  $\xi_1$  large and  $\xi_2 = 0$  to conclude that  $\mathbf{Q}(\xi_1, \xi_2) - \tilde{\mathcal{T}}$  is non-degenerate. After fixing  $(\lambda_4, \lambda_5), (\xi_1, \xi_2)$ , it is easy to see (5.7) can be solved with  $\mathbf{a} = O(\varepsilon)$ .  $\square$

## 6 Proof of Theorem 1.1 under Assumption (iii)

We are left to prove the theorem for  $N_2 \sum_{i=1}^{N_1} p_i = N_1 \sum_{j=1}^{N_2} q_j$ ,  $N_1 \neq N_2$  and one of  $N_i$  is 1. Without loss of generality, assume  $N_1 = 1$  and  $\sum_{i=1}^{N_1} p_i = \sum_{j=1}^{N_2} q_j = 0$ . In this case, for the improvement of approximate solution in the  $O(\varepsilon^2)$  term, we can not solve equation (2.30) in Section 2.4. Instead of solving (2.30), we can find a unique solution of the following equations which is guaranteed by Lemma 2.2:

$$\begin{cases} \Delta\psi_1 + |z|^{2N_1} e^{2\tilde{U}_{1,0} - \tilde{U}_{2,0}} (2\psi_1 - \psi_2) = 2|z|^{4N_1} e^{4\tilde{U}_{1,0} - 2\tilde{U}_{2,0}} - |z|^{2(N_1 + N_2)} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}} \\ \Delta\psi_2 + |z|^{2N_2} e^{2\tilde{U}_{2,0} - 3\tilde{U}_{1,0}} (2\psi_2 - 3\psi_1) = 2|z|^{4N_2} e^{4\tilde{U}_{2,0} - 6\tilde{U}_{1,0}} - 3|z|^{2(N_1 + N_2)} e^{\tilde{U}_{2,0} - \tilde{U}_{1,0}}. \end{cases} \quad (6.1)$$

We use this unique solution as the new  $\psi_0$ , and proceed as before. Then by checking the previous proof, we can get that in this case, the error  $\|E\|_* \leq C_0$  and we can get a solution  $v$  of (3.1) which satisfies

$$\|v\|_* \leq C_0, \quad (6.2)$$

for some positive constant  $C_0$ , and the following estimates hold:

$$\int_{\mathbb{R}^2} (N_{11}(v) + N_{12}(v))Z_{i,1}^* + (N_{21}(v) + N_{22}(v))Z_{i,2}^* dx = O(\varepsilon), \quad (6.3)$$

for  $i = 3, \dots, 14$ .

Then the reduced problem we get is

$$\mathbf{Q}(\mathbf{a}) + \tilde{\mathcal{T}}(\mathbf{a}) + O((1 + |\xi|)|\mathbf{a}|^2) + O(1) + O(\varepsilon) = 0, \quad (6.4)$$

where the  $O(1)$  term comes from the  $O(1)$  term of the error  $E$  since we use the solution of (6.1) instead of (2.30) as the  $O(\varepsilon^2)$  improvement. Recalling that  $\mathbf{Q} = \mathbf{Q}(\xi_1, \xi_2)$  depend on two free parameters  $\xi_1, \xi_2$  and arguing as before, we can choose  $\xi_1$  large enough. Then it is easy to get a solution of (6.4) with  $\mathbf{a} = O(\xi_1^{-\alpha})$  for any  $0 < \alpha < 1$ .  $\square$

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## References

- [1] W.W. Ao, C.S. Lin and J.C. Wei, On non-topological solutions of the  $A_2$  and  $B_2$  Chern-Simons system, *Memoirs of Amer. Math. Soc.*, to appear.
- [2] W.W. Ao, C.S. Lin and J.C. Wei, On Toda system with Cartan matrix  $G_2$ , Preprint.
- [3] L. Caffarelli and Y.S. Yang, Vortex condensation in the Chern-Simons Higgs model: An existence theorem, *Commun. Math. Phys.* 168(1995), 321-336.
- [4] D. Chae and O.Y. Imanuvilov, The existence of non-topological multi-vortex solutions in the relativistic self-dual Chern-Simons theory, *Comm. Math. Phys.* 215(2000), 119-142.
- [5] H. Chan, C.C. Fu and C.S. Lin, Non-topological multi-vortex solution to the self-dual Chern-Simons-Higgs equation, *Comm. Math. Phys.*, 231(2002), 189-221.
- [6] J.L. Chern, Z.-Y. Chen and C.S. Lin, Uniqueness of topological solutions and the structure of solutions for the Chern-Simons system with two Higgs particles, *Comm. Math. Phys.*, 296(2010), 323-351.
- [7] K. Choe, N. Kim and C.S. Lin, Existence of self-dual non-topological solutions in the Chern-Simons-Higgs model, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(2011), 837-852.
- [8] K. Choe, N. Kim and C.S. Lin, Self-dual symmetric nontopological solutions in the  $SU(3)$  model in  $\mathbb{R}^2$ , Preprint.

- [9] A. Doliwa, Holomorphic curves and Toda systems, *Lett. Math. Phys.* 39(1997),21-32.
- [10] G. Dunne, Mass degeneracies in self-dual models, *Phys. Lett.*, B 345(1995), 452-457.
- [11] G. Dunne, Self-dual Chern-Simons Theories. *Lecture Notes in Physics*, vol. m36(1995), Berlin-New York, Springer-Verlag.
- [12] G. Dunne, Vacuum mass spectra for  $SU(N)$  self-dual Chern-Simons-Higgs, *Nucl. Phys.*, B 433(1995), 333-348.
- [13] P. Esposito, M. Del Pino and M. Musso, Nondegeneracy of entire solutions of a singular Liouville equation, *Proc. Amer. Math. Soc.* 140(2012), 581-588.
- [14] H.Y. Huang and C.S. Lin, On the entire radial solutions of the Chern-Simons  $SU(3)$  system, *Comm. Math. Phys.*, to appear.
- [15] H.Y. Huang and C.S. Lin, Classification of the entire radial self-dual solutions to non-abelian Chern-Simons systems, preprint.
- [16] J. Hong, Y. Kim and P.Y. Pac, Multivortex solutions of the Abelian Chern-Simons theory, *Phys. Rev. Lett.*, 64(1990), 2230-2233.
- [17] R. Jackiw and E.J. Weinberg, Self-dual Chern-Simons vortices, *Phys. Rev. Lett.*, 64(1990), 2234-2237.
- [18] H. Kao and K. Lee, Selfsual  $SU(3)$  Chern-Simons-Higgs systems, *Phys. Rev.*, D 50(1994), 6626-6635.
- [19] K. Lee, Relativistic nonabelian Chern-Simons systems, *Phys. Lett.*, B 225(1991), 381-384.
- [20] K. Lee, Selfdual nonabelian Chern-Simons solitons, *Phys. Rev. Lett.*, 66(1991), 553-555.
- [21] C.S. Lin, J. Wei and D. Ye, On classification and nondegeneracy of  $SU(n+1)$  Toda system with singular sources, *Inventiones Mathematicae* 190(2012), no.1, 169-207.
- [22] C.S. Lin, S. Yan, Bubbling solutions for relativistic abelian Chern-Simons model on a torus, *Comm. Math. Phys.*, 297(2010), 733-758.
- [23] C.S. Lin, S. Yan, Bubbling solutions for the  $SU(3)$  Chern-Simons model on a torus, *Comm. Pure Appl. Math.* 66(2013), 991-1027.
- [24] A. Malchiodi, D. Ruiz, New improved Moser-Trudinger inequalities and singular Liouville equations on compact surfaces, *Geom. Funct. Anal.*, 21-5 (2011), 1196-1217.
- [25] A. Malchiodi, D. Ruiz, A variational analysis of the Toda-system on compact surfaces, *Comm. Pure Appl. Math.* 66(2013), no.3, 332-371.
- [26] M. Nolasco, G. Tarantello, Double vortex condensaties in the Chern-Simons-Higgs theory, *Calc. Var. Par. Diff. Eqns.*, 9(1999), 31-94.

- [27] M. Nolasco, G. Tarantello, Vortex condensates for the  $SU(3)$  Chern-Simons theory, *Comm. Math. Phys.*, 213(2000), 599-639.
- [28] J. Prajapat and G. Tarantello, On a class of elliptic problems in  $\mathbb{R}^2$ : symmetry and uniqueness results, *Proc. Royal Society Edin.* **131A** (2001), 967-985.
- [29] J. Spruck and Y. Yang, Topological solutions in the self-dual Chern-Simons theory: Existence and approximation, *Ann. Inst. H. P.*, 12(1995), 75-97.
- [30] J. Spruck and Y. Yang, The existence of non-topological solutions in the self-dual Chern-Simons theory, *Comm. Math. Phys.*, 149(1992), 361-376.
- [31] G. Tarantello, Multiple condensate solutions for the Chern-Simons Higgs theory, *J. Math. Phys.*, 37(1996), 3769-3796.
- [32] G. Tarantello, Uniqueness of self-dual periodic Chern-Simons vortices of topological type, *Calc. Var. Par. Diff. Eqns.*, 29(2007), 191-217.
- [33] R. Wang, The existence of Chern-Simons vortices, *Comm. Math. Phys.*, 137(1991), 587-597.
- [34] J. Wei, CY Zhao and F. Zhou, On non-degeneracy of solutions to  $SU(3)$  Toda system, *CRAS.* 349(2011), no.3-4, 185-190.
- [35] G. Wang and L. Zhang, Non-topological solutions of the relativistic  $SU(3)$  Chern-Simons Higgs model, *Comm. Math. Phys.*, 202(1999), no.3, 501-515.
- [36] Y. Yang, *Solitons in field theory and nonlinear analysis*, Springer Monographs in Mathematics, Springer, New York, 2001.
- [37] Y. Yang, The relativistic non-Abelian Chern-Simons equations, *Comm. Math. Phys.*, 186(1997), 199-218.